Introduction to Special Relativity
Lecture Notes

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Contents

1 Galilean Relativity and the Ether

In 1905 Albert Einstein published his classic paper *On the Electrodynamics of Moving Bodies*. This was the birth of special relativity, although it was not given that name at the time. By the end of this course you should be able to follow much of the kinematical part of this paper (or at least the English translation of it - see the link on Moodle).

Einstein's ideas were quickly built on by Hermann Minkowski and others and, despite special relativity's apparently counter-intuitive predictions, it almost immediately gained acceptance as the correct physical theory describing the relationship between space and time. Together with the later theory of general relativity, which we will not study in this course, it has since been experimentally verified to an extraordinary degree of accuracy.

We will follow a quasi-historical approach to learning the theory, beginning in this lecture by exploring the classical notions of relativity and the Galilean transformations between reference frames. You need to be very comfortable dealing with transformations between reference frames before moving on. We will then discover a fundamental incompatibility between these ideas and the electromagnetic theory which was developed in the mid-19th century. Attempts to resolve this discrepancy by introducing the idea of the 'ether' failed, leading us to the derivation of the Lorentz transformations, to the discovery of time dilation and length contraction, and to a breakdown in our notions of absolute simultaneity.

In later lectures we will encounter the more modern idea of 'spacetime' and the invariant spacetime interval. We will discover that we must modify our definitions of momentum and energy, and we will show that there is an equivalence between energy and mass, leading to perhaps the most famous physics equation of all: $E = mc^2$.

1.1 Frames of Reference and Events

A frame of reference can be thought of, informally, as a '*point-of-view from which a measurement is made by an observer*'. We will study how measurements made by observers in different reference frames are related to one another. The measurements will be, initially, of the locations and times at which events occur, and the space and time intervals between the events.

An **event** is something that happens at a particular point in space and time (which we call its co-ordinates). Observers in different reference frames will measure different co-ordinates for the same event.

We will deal mostly with non-accelerating reference frames, i.e. observers who are not accelerating. These non-accelerating reference frames are called **inertial reference frames**.

An **inertial frame of reference** is a frame of reference in which the laws of mechanics hold good, i.e. a body remains at rest or moves with a constant velocity unless acted upon by a force.

We can tell we are in an inertial reference frame because we don't feel 'pseudo forces'. For example, think of when you are in an accelerating car, and you seem to feel a force pushing you back onto the seat (sometimes known as 'G-forces'). In inertial reference frames we don't have these extra forces, the simple laws of mechanics hold; whenever we see an acceleration we also see a force that causes that acceleration ($F = ma$).

A person in an inertial frame of reference, i.e. moving at a constant speed, cannot tell that they are moving except by reference to other objects. That is, **no experiment can tell you what your absolute velocity (or speed) is**, only your velocity or speed relative to something else. For example, if you are standing on the ground and looking at a train going past, you will measure it to have a speed of u in a certain direction. But from the point of view (the frame of reference) of someone on the train, they are not moving, and it is the Earth and you moving at a speed of $-u$ relative to them. This is the Principle of Relativity, which pre-dates the theory of special relativity by several hundred years.

The **Principle of Relativity** states that no experiment can determine the absolute velocity of a frame of reference. Only relative velocities between different frames of reference can be measured.

It's also useful to define what we mean by a 'rest frame' at this point.

The **rest frame** of an object is the frame of reference in which it is not moving (i.e. its velocity is zero).

1.2 Transformations between Inertial Reference Frames

If we measure the co-ordinates of an event in two different reference frames, we can convert between those measurements using **transformations**.

Let's return to our example of a train passing by someone standing by the side of the tracks. For the driver and the observer we will define two reference frames and co-ordinate systems. We could define these in all sorts of ways, and of course it wouldn't change the physics, but there is particular way of defining things which makes the problems we will encounter as simple as possible.

The scheme we will use is shown in Figure 1.

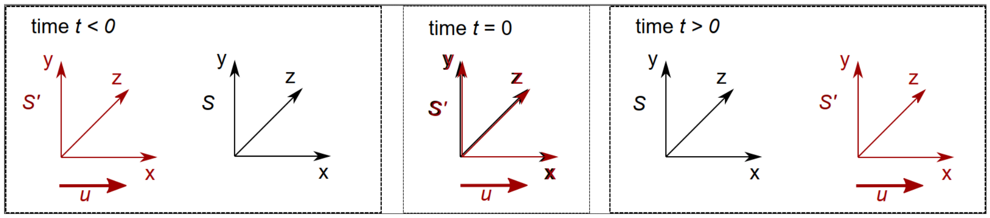

Figure 1: Illustration of two frames of reference S and S', both Cartesian co-ordinate systems. S' is moving with speed u along the positive x-axis of frame S. Three situations are shown. On the left, at time $t < 0$, frame S is to the left of frame S'. In the middle, at time $t = 0$, the origins of the two frames coincide. On the right, at $t > 0$, frame S' is to the right of S.

We call the reference frames, and their associated co-ordinate systems, S for the driver and S' (which we say 'S prime') for the trackside observer. We will measure the location and time at which events occur from both reference frames. For the location of the event, we will use standard Cartesian co-ordinate systems:

- x, y, and z for the driver's frame of reference, S;
- x', y' and z' for the trackside observer's frame, S'.

We will align the two co-ordinate frames, so that x is parallel with x' and so on. The motion of the train, and hence the S' reference frame, is along the x-axis of frame S, in the positive direction, with speed u.

We will also have co-ordinates for the time of the event, which we will call t and t' for the two reference frames. Again, for simplicity, we'll set time $t = 0$ and $t' = 0$ at the point where the train passes the trackside observer. In Newtonian physics, everyone experiences time the same, so for now we have that $t = t'$ at all times anyway, but later on we will find something quite different.

Now let's imagine some event occurring, the driver sneezing for example. Recall that an event is something that can be described by space and time co-ordinates - it happens at a certain place at a certain time. An event does not *belong* to a reference frame; it can be *measured or observed* from different reference frames.

To define the co-ordinates of this event we need to know its location in space and time (i.e. where and when it happened). In the co-ordinate system of the train driver, that is straightforward. For simplicity we will say the driver is at $(x', y', z') = (0, 0, 0)$, i.e. at the origin of his reference frame, which we call S'.

Now, what is the spatial location and time of the sneeze (the 'event') from the point of view (i.e. in the rest frame) of the observer by the side of the track?

- The train is moving past at speed u, so after an amount of time t the car will have moved ut along the x-axis.
- So, the x co-ordinate of the event will simply be $x' + ut$.
- The train has not moved along the y and the z-axes, and so $y = y'$ and $z = z'$.
- Time is the the same for everyone, $t = t'$.

We can now write down the full set of equations to transform the co-ordinates of event measured in frame S into the co-ordinates measured in frame S' and vice versa. These are known as the Galilean transformations.

$$x' = x - ut \tag{1}$$
$$y' = y \tag{2}$$
$$z' = z \tag{3}$$
$$t' = t \tag{4}$$

$$x = x' + ut' \tag{5}$$
$$y = y' \tag{6}$$
$$z = z' \tag{7}$$
$$t = t' \tag{8}$$

Notice that when going from S to S' we have a $-ut$ rather than a $+ut'$ or $+ut$. This is because, from the point of view of the train driver, the observer is moving backwards along the x-axis. Otherwise the equations are the same. It is a requirement of the principle of relativity that the transformations going from S to S' are the same as those going from S' to S (except for the sign change).

Train and Tracks Example: *A train passes a man standing by the tracks at a velocity of 10 m/s. 10 s after it passes him, the train sounds its horn. How far away does the man by the tracks measure the horn to be when it sounds?*

Solution: Let S be man's frame, S' be the train frame.

In $S: x = ?,\ t = ?$

In S': $x' = 0$ m, $t' = 10$ s

Therefore:

$$\begin{aligned} x &= x' + ut' && (9)\\ &= 0 + (10\,\mathrm{m/s})(10\,\mathrm{s}) && (10)\\ &= 100\,\mathrm{m} && (11) \end{aligned}$$

Now let's imagine that the driver of the train leans out of the window and throws a ball along the x-axis, with a speed v' relative to himself and the train. How fast does the observer think the ball is moving (v)?

Solution You can probably guess straight away that she sees the speed of the ball added to the speed of the train, i.e. $v = v' + u$, where we remember that u is the speed of the train relative to the track-side observer. Thinking of this in terms of the speeds along the x, y and z axes, we have the Galilean velocity (speed) transformations:

$$v'_x = v_x - u \tag{12}$$
$$v'_y = v_y \tag{13}$$
$$v'_z = v_z \tag{14}$$

$$v_x = v'_x + u \tag{15}$$
$$v_y = v'_y \tag{16}$$
$$v_z = v'_z \tag{17}$$

These can also be obtained simply by differentiating the Galilean transforms with respect to time, which is left a simple exercise. Notice that both observers measure the same y and z components of the velocity. The x-component is special because that is the direction of the relative motion of the two observers.

Ball on a Train Example: The train driver from the same train as above (moving at 10 m/s) throws a ball out of the window along the direction of the train's motion at 5 m/s relative to himself. What speed does the man at the side of the track measure the ball to have?

Solution: Let S be man frame, S' be train frame.

In S: v_x is unknown.

In S': $v'_x = 5$ ms^{-1}

Therefore:

$$v_x = v'_x + u \tag{18}$$
$$= 5 \text{ ms}^{-1} + 10 \text{ ms}^{-1} \tag{19}$$
$$= 15 \text{ ms}^{-1} \tag{20}$$

1.3 The Speed of Light

It has been known since the experiments of Ole Römer in the 17th century that light travels with a finite speed. Successive measurements over the years have succeeded in fixing the value in a vacuum, known as c, at approximately 3×10^8 m/s. But what is this relative to?

As an analogy, we know that sound travels in air at approximately 340 m/s, and much faster in other, denser materials. Sound is a wave in a medium. So it is reasonable that the velocity is relative to the medium (e.g. air) and indeed this is what we find experimentally. The speed of sound does not depend on the speed of the source of the sound, and so if the source is travelling through the air faster than the speed of sound then it's possible for it to overtake sound - this is the effect responsible for the sonic boom heard from supersonic jets. Conversely, inside a sealed plane, the air is moving along with the plane, so sound will continue to travel at $340\,\mathrm{m/s}$ relative to the passengers.

Unlike for sound, where we have an obvious medium supporting the sound wave, it is not clear what light is a wave *in*, since light travels through a vacuum.

In 1862, James Clark Maxwell discovered that light is in fact an electromagnetic wave - a wave in electric and magnetic fields. He showed that the speed of light can be predicted from the fundamental equations describing electricity and magnetism, known as Maxwell's Equations. Crucially, when deriving the wave equation, no account is taken of relative velocities, and we don't define a carrying medium. Maxwell's equations, unlike Newton's laws, do not appear to remain the same when moving between reference frames using the Galilean transformations. So it would be possible to determine we are moving by making measurements of electrical and magnetic phenomena. In particular, our 'absolute speed' could be determined simply by measuring the speed of light in our reference frame. For example, if we measure the speed of light to be $c + 10\,\mathrm{m/s}$ that means we are moving at $10\,\mathrm{m/s}$ relative to this absolute reference frame.

1.4 The Ether

If an absolute reference frame were to exist (we will soon see it does not), what form would it take? One idea was that there is some medium called the ether which fills the universe, and it would be this is the material in which light waves travel. The ether, if it existed, would allow light waves to travel at very high speed but also seems to be undetectable - we cannot feel it and it seems not to slow objects down that are moving through it. It does, however, apparently solve the problem of the relativity of the speed of light: the speed of light is fixed relative to the ether.

Unfortunately, as we will now see, the idea of the ether doesn't work, or rather, it contradicts the results of experiments in ways that are seemingly impossible to resolve.

1.5 Experiments to Detect the Ether

In the second half of the nineteenth century, a number of experiments were performed with the intention to detect the presence of the ether, or more specifically our motion through it. We shall discuss only one, the Michelson-Morley experiment, as it is both simple to understand and definitive.

The assumption behind the experiment is that, if the Earth is moving relative to the ether, this would mean that we would measure different speeds of light depending on the relative speed of the Earth. If we measure the speed of light along the direction of the Earth's motion, we should get a different value than if we measure the speed of light moving perpendicular to the Earth's motion. Furthermore, since the Earth is in orbit around the Sun, we would expect that the speed of the Earth relative to the ether changes throughout the year. If at one point in the year the Earth is moving at $v\,\mathrm{m/s}$, then six months later, on the other side of the orbit, it will be moving at $-v\,\mathrm{m/s}$. So if we measure the speed of light relative to the Earth, we should obtain different values.

This was the idea behind the Michelson-Morley experiment. Clearly it is quite difficult to measure the speed of light because it is so fast. However, it can be done (and could be even in 1895) with a technique called interferometry, as illustrated in Figure 2. This idea is to split light into two arms of the interferometer. Light bounces of a mirror in each arm and is then recombined. Light is a wave and so, providing the lengths of the two arms are the same, we see interference effects, giving rise to light and dark fringes. These fringes are very sensitive to the travel time of the light (either due to a change in its speed or a change in the length of one of the arms).

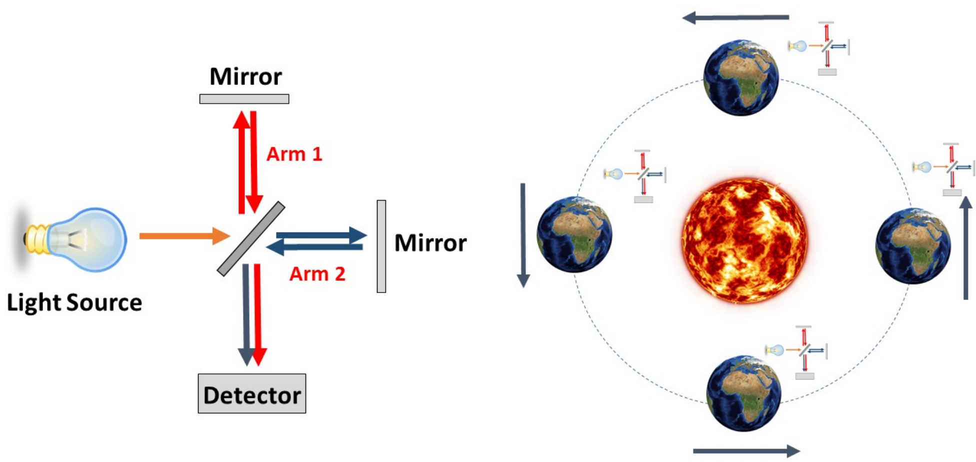

Figure 2: Illustration of the Michelson-Morley experiment. A simplified diagram of the interferometer is shown in the left; light is split by a beam-splitter into two perpendicular arms of the interferometer. In each arm there is a mirror, sending back light the way it came. The two light beams are then recombined and sent to a screen or a detector to allow interference fringes to be seen, On the right, the Earth is shown at four positions equally spaced in its orbit of the Sun, where it should have four different directions of velocity with respect to the non-moving ether, indicated by arrows.

The idea then is to set up the interferometer and observe the fringes. Then, rotate the interferometer by 90 degrees and look at the fringes again. If the speed of light is different in the two directions, the fringes will change. To avoid the possibility that the experiment happened to be carried out at the point in the Earth's orbit where it had zero speed relative to the ether, it can be repeated six months later.

No difference in the speed of light was measured. Regardless of which way around the interferometer was, or what time of year the measurements were made, the speed of light was always measured to be the same. To put it another way, light does not follow the rules for velocity transformations, *in every reference frame light appears to travel at* c!

How could this be explained? One possibility was that the ether is somehow dragged along with the Earth on its orbit, so that the experiment was always measuring the speed of light relative to the moving ether. On practical grounds it becomes hard to explain this, and it also contradicts other observations (such as stellar aberration, which we will not discuss).

A much more elegant solution, dispensing with the idea of the ether altogether, was presented in 1905 by Albert Einstein: the special theory of relativity.

2 The Lorentz Transformations

In this lecture we will resolve the problem we discovered in Lecture 1 in applying the Galilean transforms to electromagnetic radiation and the speed of light. We will do this by modifying the transformations in such a way that they reproduce the familiar classical results at low, everyday velocities, but also allow for the observation that light appears to travel at the same velocity in every reference frame.

2.1 The Postulates of Special Relativity

Einstein proposed two postulates. (Postulates are things that we *assume* are true as the basis for a theory which we then compare with experiment). The postulates are sometime written in slightly different ways, but in essence are:

1. The laws of physics are the same in all inertial frames, one cannot detect one's absolute velocity by conducting any experiment.
2. The speed of light is independent of the source, and is constant in all reference frames.

Together, these statements are not compatible with the Galilean velocity transformations stated in Lecture 1. Consider the following: I am travelling on a train at $50\,\mathrm{m/s}$, and you are standing by the side of the track. If I throw a ball forwards along the train at $10\,\mathrm{m/s}$ you will say the speed of the ball is $50 + 10 = 60\,\mathrm{m/s}$. This is simply applying the velocity transformations.

But now let's say I do the same thing, except rather than throwing a ball, I shine light from a torch along the direction the train is moving in. The second postulate says that, from my point of view, the speed of light is equal to c. If we apply the velocity transforms, the speed you measure would be $(c+50)\,\mathrm{m/s}$. But the second postulate says that you also measure the speed of light to be equal to c. Therefore, if we accept the postulates, then we need to modify the Gailean transformations.

We stated the Galilean transformations between two inertial frames of reference, S and S', to be:

$$x' = x - ut \iff x = x' + ut \tag{21}$$
$$y' = y \tag{22}$$
$$z' = z \tag{23}$$
$$t' = t \tag{24}$$

We know that these equations work well for most objects that we interact with, so the modified form of the equations must be such that they still give (approximately) the same answer for situations where we know the Galilean transformations work.

2.2 Statement of the Lorentz Transformations

The correct transformations between reference frames are called the Lorentz transformations. They had been discovered before Einstein's paper by Lorentz as a somewhat *ad hoc* solution to the problem of the Michelson-Morley experiment, although Lorentz still very much believed in the ether. The transformations are:

$$x' = \gamma(x - ut) \tag{25}$$
$$y' = y \tag{26}$$
$$z' = z \tag{27}$$
$$t' = \gamma(t - ux/c^2) \tag{28}$$

$$x = \gamma(x' + ut') \tag{29}$$
$$y = y' \tag{30}$$
$$z = z' \tag{31}$$
$$t = \gamma(t' + ux'/c^2) \tag{32}$$

where

$$\gamma = \sqrt{\frac{1}{1 - u^2/c^2}} \tag{33}$$

The γ term is the 'Lorentz factor', often simply called **gamma**. Sometimes, we also define $\beta = u/c$ to simplify equations, such that $\gamma = \sqrt{1/(1 - \beta^2)}$.

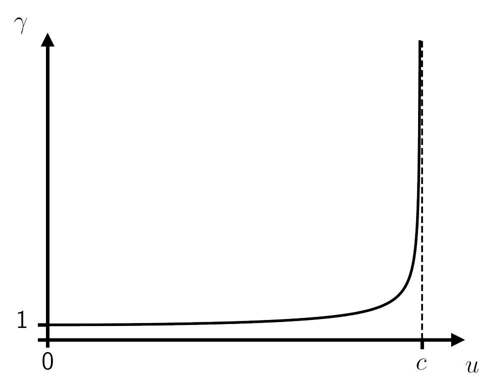

Figure 3: Plot of γ as a function of speed, u.

Notice that to switch between the two sets of equations (primed to non-primed and non-primed to primed) is simply a matter of replacing u with $-u$. That is to be expected, after all there is nothing special about which set of co-ordinates we call S and which we call S'.

We can check that the Lorentz transformation do indeed reduce to the Galilean transformations for most situations within our experience. The key thing to notice is the u^2/c^2 in the gamma term (and also the ux/c^2 in the equations for t and t'). c is a very large number, 3×10^8 m/s, whereas most velocities we deal with tend to be quite small.

For example, if we go back to our 50 m/s train, u^2/c^2 is of the order of 10^{-14}. So if we calculate γ:

$$\gamma = \sqrt{\frac{1}{1 - u^2/c^2}} = \sqrt{\frac{1}{1 - 10^{-14}}} \approx 1 \tag{34}$$

So if $\gamma \approx 1$, then $x' \approx x - ut$, which is the same as the Galilean transformation. More generally we can say that $\gamma \to 1$ for $u \ll c$. Similarly, for the t transformation, assuming $ux \ll c^2$ we similarly have $t \approx t'$. So, at the kinds of speeds we typically experience, the Galilean and Lorentz transformation give essentially the the same result. We therefore only need to use the more complicated Lorentz transformations when we have speeds are some reasonable fraction of c.

2.3 Derivation of the Lorentz Transformations

We now show that the Lorentz transformations are the only transformations that are consistent with the two postulates of special relativity.

We will assume that space is isotropic and homogeneous. That is to say that the laws of physics do not depend on the direction we are moving in or where we are in space. If that is the case then we can write the most general possible transformations between two inertial frames as

$$x' = \gamma(x - ut) \tag{35}$$
$$x = \gamma(x' + ut') \tag{36}$$

Note that we are calling the constant γ, but at this point in the derivation we don't know what γ is yet!

Why must they take this form? They must be linear, i.e. they cannot contain some power of x or t. If they contained a power of x then this would violate homogeneity. This means that γ cannot be a function of x (although it may be a function of u). There cannot be a constant term since this would violate isotropy. And for the same reason the two transforms (i.e. for x and x') must be identical except for exchanging u with $-u$.

Prior to the development of special relativity we would have said that time is measured the same in both frames of reference, and hence $t = t'$. If we substitute that into the equations above, the only way to make it work is if $\gamma = 1$. This gives us the Galilean transformations.

But now let's abandon the constraint that $t = t'$ and see what happens. Moving to the second postulate, we have that light travels at the same speed in both S and S'. So if we imagine a pulse of light moving in S at c the distance it will travel in time t is simply given by $x = ct$. In reference frame S' we similarly have that $x' = ct'$. Notice that there is no c', the second postulate says that light travels at c in all reference frames (i.e. $c' = c$).

Now, if we substitute $x = ct$ and $x' = ct'$ into our general transformations, we have:

$$ct' = \gamma(ct - ut) = \gamma t(c - u)ct = \gamma(ct' + ut') = \gamma t'(c + u) \tag{37}$$

From the second equation we have

$$t = \gamma t' \frac{(c + u)}{c}. \tag{38}$$

Substitute this into the first equation and we have

$$ct' = \gamma \frac{[\gamma t'(c+u)](c-u)}{c}. \tag{39}$$

The t' on both sides cancel, and we can multiply both sides by c to get

$$c^2 = \gamma^2 (c+u)(c-u). \tag{40}$$

Now, $(c+u)(c-u)$ is equal to $c^2 - u^2$ and so we have

$$\gamma^2 = \frac{c^2}{c^2 - u^2}. \tag{41}$$

So, dividing through the right hand side by c^2 and taking the square root we have

$$\boxed{\gamma = \sqrt{\frac{1}{1 - u^2/c^2}}} \tag{42}$$

which is exactly the γ factor in the Lorentz transformations.

We can then obtain an expression for t' by substituting $x' = \gamma(x - ut)$ into $x = \gamma(x' + ut')$ giving

$$x = \gamma^2 x - u\gamma^2 t + u\gamma t'. \tag{43}$$

From this point you could solve this directly for t' or t by substituting in the expression for γ. Alternatively, we can rearrange to obtain

$$t' = \gamma t - x\frac{\gamma^2 - 1}{u\gamma}. \tag{44}$$

We then need to know the following identity, which can be found from the definition of γ,

$$\gamma^2 - 1 = \frac{u^2}{c^2}\gamma^2. \tag{45}$$

Substituting in this expression for $\gamma^2 - 1$ leads directly to

$$\boxed{t' = \gamma\left(t - \frac{ux}{c^2}\right)} \tag{46}$$

which again is what we want. We have therefore shown that the Lorentz transformations are the **only** possible transformations consistent with the postulates of special relativity.

2.4 Working with the Lorentz Transformations

If we know the space and time co-ordinates of an event in one reference frame S (x and t) then we can use the Lorentz transformations to calculate its space and time co-ordinates (x' and t') in another reference frame, S', providing we know the velocity, u which S' is moving with relative to S. Bear in mind that, when not otherwise stated, we are assuming that we are working in our standard co-ordinate system, where the two observers agree that $t = t'$ at the instant when the zero of the S' frame moves past the zero of the S frame.

Another factor to keep in mind is the light travel time. When we talk about the t co-ordinates of an event being measured by an observer, we don't mean the time at which an observer would see the event if they were looking at it through a telescope, This is because it would take some amount of time for the light to travel from the event to the observer and her telescope. This is *not* what the Lorentz transformations are telling us about. The Lorentz transformations tell us what the observer would measure even if they corrected their measurement to allow for the travel time of the light. To put it another way, special relativity is not telling us about some apparent change in time due to the time it takes light to travel, but a real physical effect.

One way this is sometimes expressed is to imagine a reference frame, S or S', as being an infinite lattice of clocks, moving rigidly together, with all the clocks perfectly synchronised. The Lorentz transformations tell us what would be measured by a clock in the lattice that is located at the event, not what an observer looking at that clock through a telescope would see.

Lorentz Transformation Problem *A spaceship departs Earth at a constant speed of $c/2$. After 1 year (by the pilot's clock) the pilot measures an explosion which is 1 light year directly ahead. According to an observer on Earth, how long after the spacecship left did the explosion occur, and how far was is from Earth?*

Solution:

- Let's define S = Earth frame and S' = spaceship frame.
- The spaceship leaves the Earth at time $t_0 = t'_0 = 0$.
- The event occurs at time t as measured on Earth and time t' as measured on the spaceship, at locations x and x', respectively.
- So in S': $x' = 1$ ly, $t' = 1$ year
- And in S': $x' = ?$, $t' = ?$
- And we have $u = c/2$

To find the time measured from Earth:

$$
\begin{aligned}
t &= \gamma(t' + ux'/c^2) && (47)\\
&= \frac{1}{\sqrt{1 - u^2/c^2}}(t' + ux'/c^2) && (48)\\
&= \frac{1}{\sqrt{1 - (1/2)^2}}(1[\mathrm{yr}] + 1/2[\mathrm{yr}]) && (49)\\
&= 2\frac{\sqrt{3}}{3}\frac{3}{2}\ \mathrm{yrs} && (50)\\
&= \sqrt{3} \approx 1.7\ \mathrm{yrs} && (51)
\end{aligned}
$$

If you're wondering where the cs went, one cancels with the c in the velocity $(c/2)$. The other converts the distance in light years to a time in years ($x' = 1\,\mathrm{ly}$ and so $\frac{x'}{c} = 1\,\mathrm{yr}$).

To find the distance measured from Earth, we similarly use:

$$\begin{aligned}
x &= \gamma(x' + ut') && (52)\\
&= 2\frac{\sqrt{3}}{3}\bigg(1\ [\text{ly}] + (c/2)(1\ [\text{yrs}])\bigg) && (53)\\
&= 2\frac{\sqrt{3}}{3}(3/2\ [\text{ly}]) && (54)\\
&= \sqrt{3} \approx 1.7\ \text{ly} && (55)
\end{aligned}$$

3 Time Dilation and Length Contraction

In this section we will determine some general consequences of the Lorentz transformations. We will show that, if I am moving relative to you, my clock will appear to you to be running slow by a factor of γ, and that my length in the direction of motion will contract by a factor of γ (i.e. divided by γ). However, from my point of view, it is *you* who is moving, and *your* clock that will run slow and *you* that will be length contracted. It might seem that this leads to paradoxes - how can your clock run slower than mine if mine is running slower than yours?. We will explore the resolution of this (and other) apparent paradoxes, and see that framework of special relativity is in fact, entirely consistent (and, as far as we know, correct).

3.1 The Light Clock

The light clock is a thought experiment which provides an introduction to time dilation without using the Lorentz transformations, and can provide an intuitive explanation of why we must abandon notions of absolute time if the postulates of special relativity are to be accepted.

Imagine a pair of mirrors, with a pulse of light bouncing between them. Each time light hits a mirror, we can think of this as the tick of a clock, and we measure the time interval between two of these *ticks*. The distance between the mirrors we call h.

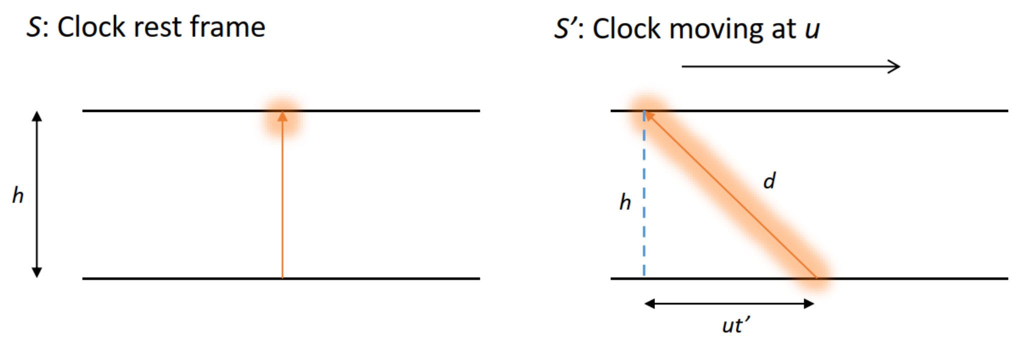

Figure 4: The light clock as observed by an observer moving with the clock (left, S) or relative to the clock (right, S'). For the observer moving with the clock, the light pulse travels directly up. For the observer who sees the clock moving, the light pulse travels diagonally.

For an observer in the frame of reference of the light clock (i.e. someone who is not moving with respect to the light clock), which we can call S, then the time taken for the pulse to pass between the mirrors is given by $t = h/c$, where c is the speed of light, as normal in classical mechanics.

An observer who is in a frame of reference moving with a velocity u with respect to the light clock sees the light travel in a zig-zag pattern, as illustrated. Let's assume they measure the time between ticks to be t'. In t' the light clock will have moved a distance of ut'. So, from Pythagoras' theorem, the pulse of light has travelled a distance d where $d^2 = h^2 + (ut')^2$. It travels this distance in a time

$t' = d/c$, and so substituting in $d = ct'$ and $h = ct$ from above we obtain:

$$c^2 t'^2 = c^2 t^2 + u^2 t'^2 \tag{56}$$
$$t'^2 (c^2 - u^2) = c^2 t^2 \tag{57}$$
$$t'^2 = t^2 \frac{c^2}{c^2 - u^2} \tag{58}$$
$$t' = t\sqrt{\frac{1}{1 - u^2/c^2}} \tag{59}$$
$$t' = \gamma t \tag{60}$$

As we will see below, this is exactly the time dilation equation.

3.2 Statement of Time Dilation

Consider a time period, $\Delta\tau$, measured in a particular frame of reference. For example, this could be the time between two ticks of a clock, as measured by someone who is not moving relative to the clock (i.e. in the same frame of reference as the clock). This is therefore a time interval between two events measured on a clock that is present at both events. We call this the **proper time**.

> The **proper time** is the time measured by a clock in its rest frame, the frame of reference in which the clock is not moving.

Now think about the point-of-view of an observer in a different reference frame who has their own clock. So for them, the original clock is moving. During a tick of the original moving clock, the time interval measured by the observer on their clock will be:

$$\Delta t = \gamma \Delta\tau \tag{61}$$

where γ is the normal gamma factor and $\Delta\tau$ is the time interval measured on the moving clock in that clock's *rest frame*. Since γ is always bigger than 1, then they will always measure a bigger time interval on their clock. So moving clocks appears to be running slow.

> Time Dilation means that, in a reference frame in which it is moving, a clock runs slow by a factor of γ compared with its rest frame.

Here we are talking about time intervals between two events - we used Δt and $\Delta\tau$. If we set up our frames of reference so that the first event is at time 0 for both observers, then the times for the second events, as measured by the observers, are related in the same way: $t = \gamma\tau$.

3.3 Time Dilation from Lorentz Equations

The time dilation equation comes quite simply from the Lorentz transformations. Imagine a bulb that regularly flashes, say at 1 second intervals. We can think of this as a simple kind of clock. Let's call the time between flashes Δt, as measured by the clock (or someone in the same rest frame of the clock, which we will call S). The first flash we will label event 1, and the second flash event 2.

In S the bulb is not moving, it is always at the same place, so we can say:

$$x_1 = x_2 = 0 \tag{62}$$

We'll call the time of the first flash $t_1 = 0$ and the second flash $t_2 = \Delta t$.

Now let's use the Lorentz transformations to figure out what someone who is moving with respect to the bulb, in frame S', sees:

$$t'_2 = \gamma(t_2 - \frac{ux_2}{c^2}) \tag{63}$$

Since $x_2 = 0$ this is simply

$$t'_2 = \Delta t' = \gamma \Delta t. \tag{64}$$

This is the same as the time dilation equation (albeit with slightly different nomenclature).

3.4 Solving Time Dilation Problems

Example: Time Dilation of a Spacecraft: *A spaceship sets out from Earth at a speed of $c/4$. The ship reaches its destination after 1 year as measured by the clock of an observer on Earth. How much time does the pilot of the spaceship measure the trip took on his clock?*

Solution: To solve this problem we can either start from the Lorentz transformations or, with a little thought, simply use the time dilation equation, $t = \gamma\tau$. First, we know that $u = c/4$ so we can calculate γ:

$$\gamma = \sqrt{\frac{1}{1 - u^2/c^2}} \tag{65}$$

$$= \sqrt{\frac{1}{1 - (1/4)^2}} \tag{66}$$

$$= 1.03 \tag{67}$$

- Define S as the Earth rest frame, S' as the spaceship rest frame.
- The first event is the spaceship leaving, which we will set to be at $x_0 = x'_0 = 0$ and $t_0 = t'_0 = 0$ (i.e. the common origin of S and S').
- The second event is the spaceship arriving. Hence $t = 1$ year and t' is the wanted unknown.
- The location of the arrival event is $x' = 0$ (since spaceship does not move in its frame of reference) and x is unknown and not needed.

Possible Lorentz transformations to use are $t' = \gamma(t - ux/c^2)$ and $t = \gamma(t' + ux'/c^2)$. Since x is unknown, use

$$
\begin{aligned}
t &= \gamma(t' + ux'/c^2) && (68)\\
&= \gamma(t' + 0) && (69)\\
&= \gamma t' && (70)\\
1 \text{ yr} &= 1.03t' && (71)\\
t' &= 0.97 \text{ yrs} && (72)
\end{aligned}
$$

If we want to start straight from the time dilation equation ($t = \gamma\tau$), we need to be careful to work out which time is t and which is τ. Recall that τ is the proper time, it is the time measured by a clock in a reference frame where the clock is not moving. However, it is very easy to get confused when working with this definition.

It is easier if we **think in terms of events**. Event 1, in this case, was the spaceship leaving Earth, and Event 2 was the spaceship arriving at its destination. For somebody on Earth, these two events are at different locations. For somebody on the spaceships these two events are at the same location. So, it is the spaceship that measures the proper time (τ). We therefore have:

$$
\begin{aligned}
t &= \gamma\tau && (73)\\
1 \text{ yr} &= (1.03)\tau && (74)\\
\tau &= 0.97 \text{ years} && (75)
\end{aligned}
$$

Another way to think about it is that, for an observer on Earth, the spaceship is moving. Therefore the observer sees their moving clock run slow, i.e it will tick up less time than the clock on Earth.

The confusion comes because the observer on the spaceship also sees the Earth's clock run slow. The question doesn't ask you about this, but you see how you can easily get it confused. **So, if in doubt, start with the Lorentz transformations**; providing you carefully define your reference frames then you can't really go wrong.

3.5 Statement of Length Contraction

Consider a rod of length L, as measured in its rest frame (i.e. by an observer who is not moving relative to the rod).

> The **proper length** is the length of an object measured in its rest frame.

For an observer in a reference frame where the object is moving, the length of the object along the direction of motion will reduce by a factor of γ:

$$
\boxed{L = \frac{L_0}{\gamma},} \qquad (76)
$$

where L_0 is the length of the object in its *rest frame*. This is called **length contraction** or Lorentz-Fitzgerald contraction.

In a reference frame in which it is moving, the length of an object reduces by a factor of γ relative to its length in its rest frame.

3.6 Demonstration of Length Contraction from Lorentz Equations

Consider an observer, in frame S', moving past a rod. We will call the rest frame of the rod S. We will synchronise clocks as the observer passes the front of the rod, and call this event 1, so that:

$$x_1 = 0, \quad t_1 = 0, \quad x_1' = 0, \quad t_1' = 0 \tag{77}$$

At exactly the same time, the moving observer measures the position of the back of the rod, x_2'. Their measurement of the length of the rod, L', is

$$L' = x_2' - x_1' = x_2' - 0 = x_2' \tag{78}$$

Since the measurements must be made at the same time

$$t_1' = t_2' = 0 \tag{79}$$

Finally, we know that in the rod's rest frame the distance between the front and back of the rod is its length, i.e.

$$L = x_2 - x_1 = x_2 - 0 = x_2 \tag{80}$$

Now we can use the Lorentz transformation:

$$x_2 = \gamma(x_2' + ut_2') \tag{81}$$

and since $t_2' = 0$, we have $x_2 = \gamma x_2'$ and hence:

$$x_2' = \frac{x_2}{\gamma}, \tag{82}$$

or

$$L' = \frac{L}{\gamma}, \tag{83}$$

as we wanted. Now, you might wonder, what would happen if we used the inverse transform

$$x_2' = \gamma(x_2 - ut_2) \tag{84}$$

If we say $t_2 = 0$ then we would have $x_2' = \gamma x_2$ and so $L' = \gamma L$! rather than $L' = \frac{L}{\gamma}$, and they can't both be right!

The answer is that we have absolutely no business saying that $t_2 = 0$. We said that in the frame of the observer, S', then $t_2' = 0$ because we were setting $t_2' = t_1'$. To measure the length of something that is moving, we have to compare the positions of the front and the back *at the same time*.

However, there is nothing to say that these two times are equal in S. You could work out $t_2 = \gamma(t_2' - ux_2')$ of course, substitute this in, and you would get the correct answer, but it's a bit more maths.

3.7 Solving Length Contraction Problems

Moving Object Example: *A train of proper length 100 m passes by an observer at a speed of $c/2$. What does the observer measure the length of the train to be?*

Solution: The proper length is 100 m, so we expect the length measured by any other observer moving relative to the train, to be less than this. In this case, $\gamma = 1.15$ and so:

$$\begin{aligned} L &= L_0/\gamma && (85)\\ &= (100\text{ m})/1.15 && (86)\\ &= 87\text{ m} && (87) \end{aligned}$$

3.8 Muon Flight Time

An experimental demonstration of the effect of time dilation on Earth comes from muons. Muons are particles created in the upper atmosphere from the decay of pions, at a height of several thousand kilometres. They are unstable, and themselves decay with a half life of around $2\,\mu$s.

A typical muon moving at $u = 0.998c$ only travels around 600 m in $2\,\mu$s. Therefore, since the distance that needs to be travelled is several times this, very few muons should reach the Earth's surface. However, we actually observe a significantly larger flux of muons reaching the Earth's surface than expected.

This is explained by time dilation. The $2\,\mu$s is a proper time interval, i.e. it is the time measured from the rest frame of the muon (the muon is at the event of the muon's creation and the event of it reaching the Earth's surface, so it must measure the proper time). From the point of view of us on Earth, however, the muons experience time dilation. At a velocity of $0.998c$, γ is about 15, meaning that the observer on earth sees the internal 'clock' of the muon seem to run 15 times slower. To put it another way, the decay half life increases by a factor of 15.

But what happens if we look at things from the rest frame of the muon. From here it is the Earth that is moving towards the muon at a speed of u. So the muon sees time running more slowly for the Earth, it obviously does not see it own clock running slow - its proper time does not change, and it is the proper time that determines the decay rate.

The answer is length contraction. In the rest frame of the muon, the Earth is moving towards it at a speed of $0.998c$. The distance that it has to travel is therefore length contracted by a factor of $\gamma = 15$. Therefore, while the muon still decays in $2\,\mu$s, now it only has to travel a distance of $L/15$ before hitting the Earth.

The fact that something that looks time dilation in one frame of reference can look like length contraction in another tells us something about the inter-connectedness of space and time, that in a sense we are looking at two different aspects of the same thing. We will explore this idea of 'spacetime' more in the next lecture.

4 Spacetime and the Relativity of Simultaneity

We will now looks at some deeper aspects of special relativity, studying the breakdown in our ideas of absolute simultaneity. Some **non-examinable** ideas are also covered, including the spacetime interval, a quantity which *can* be considered absolute in the sense that it does not change under a Lorentz transformation. This is include here for interest, you will study it in more detail in later courses.

4.1 Ladder Paradox

The ladder "paradox" is a classic situation in special relativity. If you understand the ladder paradox you are a good way to understanding the key ideas of the theory.

A farmer is standing next to his shed. He would like to put his ladder inside the shed, but he has a problem. The shed is 2 m in length while the ladder is 4 m long. Luckily, he has studied PH304, and he has an idea. He realises that if he moves the ladder at a velocity of $\sqrt{3}c/2$, he will have a gamma factor of $\gamma = 2$. The ladder will then be length contracted, in the frame of the farmer and the non-moving shed, by a factor of 2, making it 2 m in length and just able to fit inside the shed.

Of course, it won't be in the shed for very long before it crashes through the other side. However, in a second stroke of luck, the shed has two sets of doors, front and back. The ladder will enter the front door, which he can then quickly close, momentarily trapping the ladder inside. To prevent any damage to his shed doors, he will then immediately open the back door, allowing the ladder to pass through and carry on its way at $\sqrt{3}c/2$.

Leaving aside questions of how he accelerates the ladder, or opens and closes the doors so fast, does this work?

From the rest frame of the shed, it looks like it will indeed work. The difficulty comes when we consider things from the rest frame of the ladder (the ladder's point of view). From this frame it is the shed that it is moving, and so length contracted by a factor of 2. So now the shed is only 1 m in length, while the ladder maintains its length of 4 m in its rest frame. So in the rest frame of the ladder, things are even worse than they were to start with!

How is this paradox resolved? The answer is that the closing of the front and back doors of the shed are events separated in space. This means that while they are simultaneous for the farmer/shed, they are not simultaneous for the ladder. The ladder will see the back door open *before* the front door closes, and so it will never be entirely within the shed.

There is a problem on Worksheet 2 in which you will work through this in more detail.

4.2 Relativity of Simultaneity

It is clear from the Lorentz transform for time:

$$t' = \gamma(t - ux/c^2), \tag{88}$$

that time is measured differently for observers moving in different inertial frames. A consequence of this, which may not be immediately apparent, is that two events which are separated in space, and which one observer considers to occur at the same time (i.e. they are simultaneous), will not be simultaneous in a different inertial frame. Indeed, in some circumstances it may even be that the two observers do not even agree on the ordering of the events. It is often the failure to fully appreciate the breakdown of absolute simultaneity which can leads to difficulties in our understanding and apparent "paradoxes".

Consider two events occurring, events 1 and 2. They occur at two positions along the x-axis, x_1 and x_2, at times t_1 and t_2, as measured by an observer in reference frame S. Let's say that in reference frame S they are simultaneous, that is $t_1 = t_2$, and let's call this time $t_1 = t_2 = 0$.

Now consider a second reference frame, S' moving at u. From the Lorentz transformations, we can calculate the times of the events in S':

$$t'_1 = \gamma(t_1 - ux_1/c^2) = \gamma(-ux_1/c^2) \tag{89}$$

and

$$t'_2 = \gamma(t_2 - ux_2/c^2) = \gamma(-ux_2/c^2) \tag{90}$$

It's immediately obvious that $t'_1 \neq t'_2$ unless $x_1 = x_2$. That is to say that events which are separated in space and are simultaneous in S are not simultaneous in *any other frame* S'. We can also see that the order of t'_1 and t'_2 is not fixed, and depends on the values of x_1 and x_2.

4.3 Twin Paradox

The twin paradox is less illustrative of the principles of special relativity, but is nevertheless important for other reasons. It is illustrated in Figure 5.

An astronaut sets out from Earth at a speed v, which is some appreciable fraction of the speed of light. She travels for some distance, turns around and comes back. The question is, will her clock agree with the clock of her twin sister who has stayed on Earth? Or to put it another way, will they have aged differently?

From the point of view of the twin on Earth, the astronaut has been moving, and so time should have been running slower for the astronaut; while the astronaut measures a time t on her clock, the twin at home measures a time γt on her clock. So when the astronaut arrives back at Earth, she will have aged less and be younger than the twin who stayed behind. So far, so good.

But now let's look at it from the point of view of the astronaut. In her rest frame, she says that it is actually the Earth that moves away and then comes back. So from her point of view, everything is reversed, and she says that it will be the twin who stayed on Earth who will be the younger.

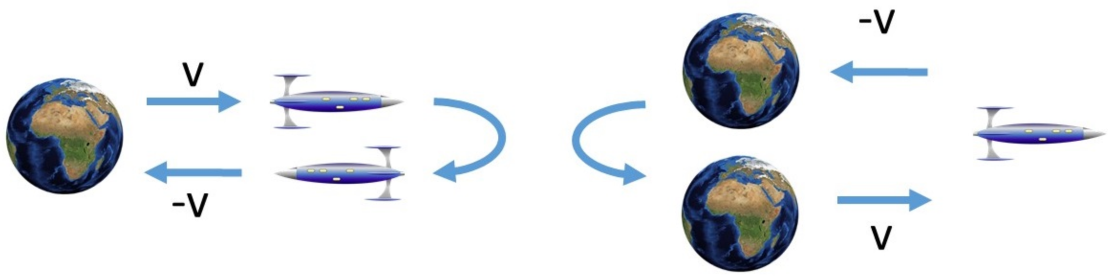

Figure 5: Illustration of the twin paradox, from the point of view of the Earth (left) and the spaceship pilot (right). From the point of view of the Earth, the spaceship goes out and comes back. From the point of view of the spaceship pilot, it is the Earth that moves away and then turns around and comes back.

Clearly both can't be correct; when they stand next to each other and compare their clocks, they cannot both be the slow ones. So what has happened?

To find the answer we return to the first postulate, which says that the laws of physics are the same in all *inertial* frames. But the astronaut was not in an inertial frame, she would have felt the acceleration as she left Earth, the deceleration at the mid-way point and acceleration back towards Earth and then a final deceleration as she arrived.

In principle we can ignore the accelerations at Earth - we could always synchronise clocks in a flypast. But there is no way to avoid the acceleration at the mid-way point. One way of thinking about it is that, at the time she turned around, the astronaut jumped between two different inertial frames travelling in opposite directions.

So who is correct? The answer is the twin who stayed on Earth. Since she stayed in an inertial frame at all times, it is perfectly valid for her to calculate everything using the Lorentz transformations. For the astronaut, it is not.

This goes to the heart of the matter of symmetry in special relativity. Two observers moving relative to each other both say that the other's clock is running slow. This is not a paradox, because the clocks are spatially separated at all times (except for time $t = 0$ for which $t' = 0$ so they both still agree.) In order to bring the clocks together, so we can look at them side by side, we require an acceleration, in which case we change the calculation!

4.4 Aside - Spacetime Diagrams [Not Examinable]

We can represent objects in space and time by a diagram called a **spacetime diagram**, shown in Figure 6. Since there are three spatial dimensions (x,y,z) plus time t, a true spacetime diagram would need to be four dimensional! However, since we are normally only interested in the position along the axis of motion, which we label as x, we can draw useful spacetime diagrams only involving x and t.

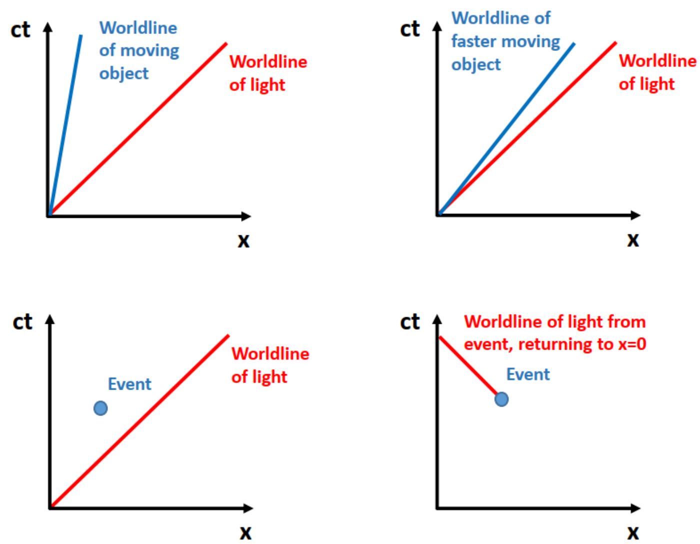

Figure 6: Four examples of spacetime diagrams. Each diagram has ct on the y-axis and x on the x-axis. The worldline of light is a line at 45 degrees from the origin. The worldline of a moving object is a line at a smaller angle than 45 degrees from the y-axis. Events are marked as a point in the diagram. The final figure shows a worldline of light from an event, returning to $x = 0$.

Individual events can be plotted, and the path an object takes through a spacetime diagram is called its **worldline**. Notice a couple of things. Firstly, we plot ct rather than t. This is so that the worldline of light is at 45 degrees. Secondly, ct is plotted on the y-axis and x on the x-axis. This may be slightly counter-intuitive, as we are used to thinking of t as the independent variable going on the x-axis, but it quickly becomes natural with practice. Try to think of x and ct as just two different co-ordinates in spacetime.

4.5 Aside - Spacetime interval - an invariant [Not Examinable]

We speak about a quantity being **Lorentz Invariant** if it does not change *when we switch between reference frames*. Note that this is a very different concept from a conserved quantity (such as momentum) which doesn't change *in time*.

Clearly x and t are not Lorentz invariant, since the Lorentz transformations tell us that they change between reference frames. But we can ask whether there is some other invariant quantity which is a combination of x and t.

An analogy which may be useful is to think of the points on a circumference of a circle. As we move around the circle, which we can think of as a rotation, our x and y co-ordinates change. However, the radius of the circle (our distance from the centre) is invariant, given simply by $r^2 = x^2 + y^2$.

There is something similar we can calculate for space and time under Lorentz transformations. While space and time (x and t) change, there is a value that we can calculate which is invariant. This invariant is called the **spacetime interval** or **spacetime distance** and is given by:

$$(\Delta s)^2 = (\Delta x)^2 - (c\Delta t)^2 \tag{91}$$

This looks a little similar to our circle invariant, although with some obvious differences. In particular, we notice that it is $-(c\Delta t)^2$ and not $+(c\Delta t)^2$. This means that the spacetime interval can be negative or positive, depending on whether $(c\Delta t)^2$ is smaller or bigger than $(\Delta x)^2$. By analogy to the circle we can think of a Lorentz transformation being something like a rotation in spacetime, with the proviso that the geometry of spacetime is a little different from the Euclidean geometry we are used to.

Whether the right-hand-side is positive or negative (or 0) has a very important physical meaning. First lets consider the very special case of $(\Delta s)^2 = 0$. This means that

$$(\Delta x)^2 = (c\Delta t)^2 \tag{92}$$

$$\Delta x = c\Delta t \tag{93}$$

If the distance travelled is c times the time, then the velocity is obviously c. We call this ***lightlike*** or *'null'*.

If two events are separated by $(\Delta s)^2 > 0$ then this means than the distance between them is larger than $c\Delta t$. We call this a ***spacelike*** interval. It means that there is some reference frame where the two events are simultaneous, in which case they will be separated by a distance of Δd (but only in that special reference frame). There can be no causal link between the two events, because even a signal travelling at the speed of light couldn't have covered the distance in time. Therefore a change in the order of the events does not violate causality - there can be no mixing of cause and effect.

If two events are separated by $(\Delta s)^2 < 0$ then we call this interval ***timelike***. There is no reference frame where these two events are simultaneous, but there is a reference frame where they occur at the same place at different times, with a time difference of $t = \sqrt{-\Delta s^2)}/c$. All observers must agree on the order in which the two events occurred. This is critical because, if two events are timelike separated, then there could have been a causal link between them, and no-one can observe an effect happening before its cause!

4.6 Aside - Intervals on Spacetime Diagrams [Not examinable]

A nice way to visualise spacelike and timelike spacetime distance is using our spacetime diagrams. If we place an observer at the origin and draw lines for $x = ct$ and $x = -ct$ we obtain what are known as 'light-cones', shown in Figure 7. For positive times this is the 'future light cone' and for negative times it is the 'past light cone'. Only events in your past light cone can influence you now, and you can only influence events in your future light cone. Signals from events which are outside your past light cone (events which are *spacelike separated* from you) cannot possibly have reached you yet, and you cannot possibly send a signal to events outside your future light cone.

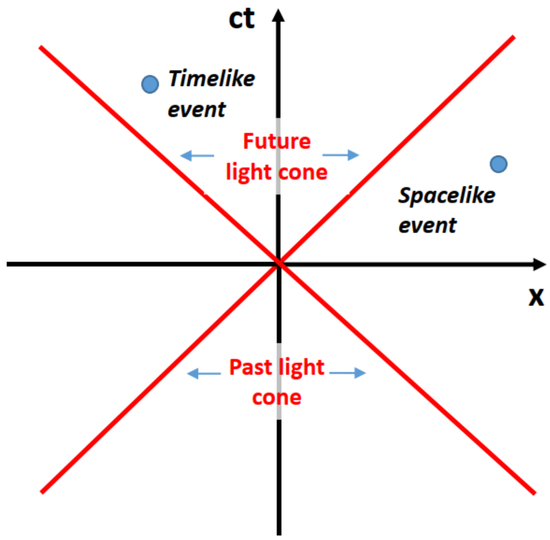

Figure 7: Past and future lightcones illustrated on a spacetime diagram. Events that are timelike separated from the origin are inside the light-cone while events that are spacelike separated are outside the light-cone.

Note that the spacetime interval will not be on the exam (although you may find it in past papers) - This is now covered, along with other invariants, as part of the Stage 2 relativity course.

5 Relativistic Doppler Effect

In this lecture we will explore the Doppler effect from a relativistic perspective and derive an expression for the wavelength shift of light due to relative motion between the source and a receiver. This has some important applications in cosmology, particularly related to the cosmological redshift.

5.1 Non-relativistic Doppler Effect

We are familiar with the Doppler effect for sound waves in everyday experience - it is the effect that causes the pitch of the noise from a car to be higher as it comes towards you, and lower as it moves away.

The explanation for this is intuitively simple when we consider that sound is a wave, travelling through air. If we are stationary (with respect to the air), then for sound waves generated by the car moving towards us the crests of the waves are closer together than they otherwise would be, and further apart when it is moving away. Closer wave crests means a higher frequency or pitch. This is illustrated in Figure 8.

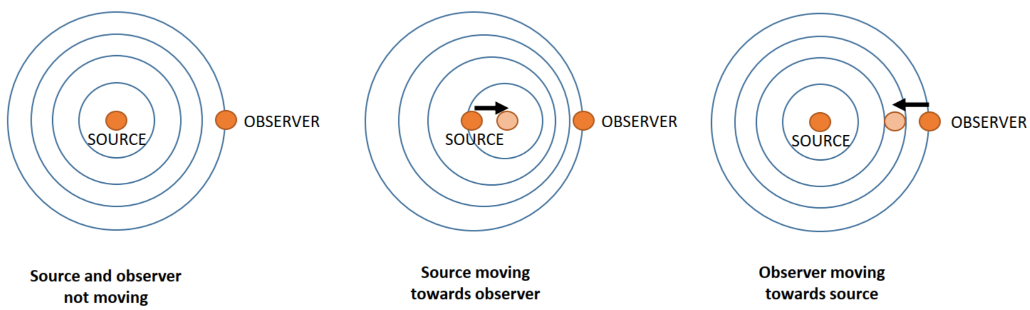

Figure 8: Illustration of Doppler effect for sound. When the source and receiver are not moving, crests of waves leaving the source arrive at the receiver at the same rate, i.e. the frequency is unchanged. When the source is moving towards the observer, each crest is created at positions that are successively closer to the receiver and so the crests are bunched together; the receiver sees a higher frequency. When the source is moving away the opposite happens, each crest is created further away, and so the crests are more spaced out when arriving at the receiver and they see a lower frequency.

Similarly, if we were moving through the air towards the car, the crests of the waves would appear closer together, giving a higher frequency, while if we were moving away, the waves would be further apart, giving a lower frequency.

To express this mathematically, consider a source of sound, generating waves such that a crest is created every Δt seconds. The waves travel at a velocity v. If the source is moving towards an observer at velocity u, then it will move a distance $\Delta x = u\Delta t$ between the creation of each wave crest. So the next crest will have to travel a distance of $u\Delta t$ less to reach the observer than if the source was non-moving. Since wave crests are travelling at v, this means the next crest will arrive a time $u\Delta t/v$ earlier than it otherwise would have.

So, if we call the time difference between each peak as measured by the observer $\Delta t'$, then:

$$\Delta t' = \Delta t - u\Delta t/v = \Delta t(1 - u/v) \tag{94}$$

Now if we remember that the frequency of a wave is the inverse of the period, $f = 1/\Delta T$, and we label the frequency seen by the observer as $f_{obs} = 1/\Delta t'$ and the frequency sent by the source as $f_{source} = 1/\Delta t$, then we have:

$$\boxed{f_{obs} = \frac{1}{1 - u/v} f_{source}} \tag{95}$$

A similar argument shows that, if the observer is moving towards the source with velocity u,

$$\boxed{f_{obs} = \left(1 + \frac{u}{v}\right) f_{source}} \tag{96}$$

These are the equations for the non-relativistic Doppler effect.

5.2 Relativistic Doppler Effect

When dealing with light, we no longer have a medium that the source and the observer can move relative to. So it can only be the velocity of the source relative to the observer than can matter.

The correct equation for the relativistic Doppler effect is:

$$f_{obs} = \sqrt{\frac{1 + u/c}{1 - u/c}} f_{source} \tag{97}$$

where u is velocity of the source relative to the observer.

5.3 Derivation of the Relativistic Doppler Effect

To derive the relativistic Doppler effect equation, we will work in the reference frame of the observer. The source is therefore moving at velocity u. If we think back to our equation for the observed time spacing between peaks arriving at the observer:

$$\Delta t' = \Delta t(1 - u/v) \tag{98}$$

then we need to make two changes to accommodate working with light. The first is that the velocity, v, is now the velocity of light, c. The second is that, since the observer sees the source in motion, we need to allow for time dilation. We therefore have

$$\Delta t' = \gamma(1 - u/c)\Delta t. \tag{99}$$

That's all the physics we need, so now we expand out the γ term to obtain:

$$\Delta t' = \frac{1-u/c}{\sqrt{1-u^2/c^2}}\Delta t \tag{100}$$

$$= \sqrt{\frac{(1-u/c)^2}{(1+u/c)(1-u/c)}}\Delta t \tag{101}$$

$$= \sqrt{\frac{1-u/c}{1+u/c}}\Delta t. \tag{102}$$

Frequency is $1/\Delta t$ and so, we obtain

$$\boxed{f_{obs} = \sqrt{\frac{1+u/c}{1-u/c}} f_{source}} \tag{103}$$

We derived this for the case that the source was moving towards the observer with velocity u. If the source is moving away, we need to use $-u$, so that $1 + u/c$ becomes $1 - u/c$ and vice-versa.

Notice something else critical here. u is the velocity of the source relative to the observer. Unlike for sound, where we cared about the velocity of both the source *and* the observer relative to the air, with light we only care about the relative velocity between the source and the observer. That is to say, we don't need to derive a second equation to tell us what happens when the observer is in motion. So in some ways, the relativistic Doppler Effect is simpler!

5.4 Relativistic Doppler Effect in Cosmology

In the cosmology section of this module, you will learn about the expansion of the universe being measured by the red shift. The idea is that if we look at wavelength spectra of light coming from stars, we see that there are lines at wavelengths corresponding to certain atomic transitions. The wavelengths that these lines should appear at are well-known from studies here on Earth. When we look at the stars, we notice that the positions of these lines appear to be shifted in wavelength. This is due to the relativistic Doppler effect, telling us that the stars are moving relative to us. (In fact they are almost always moving away, but that is another discussion).

As an example, consider measuring the spectrum from a star. We know that there should be a highly characteristic dark line in the spectrum at a wavelength of 479.8 nm, which is due to the presence of hydrogen (called the H-alpha line). However, when we observe the star we instead see the line at 537.4 nm. How fast must the star be moving?

First we need to remember that $\lambda = c/f$ where λ is wavelength and f is frequency. So if we rewrite the Doppler equation in terms of wavelength, we have:

$$\lambda_{obs}/c = \sqrt{\frac{1-u/c}{1+u/c}}\lambda_{source}/c \tag{104}$$

$$\lambda_{obs} = \sqrt{\frac{1-u/c}{1+u/c}}\lambda_{source} \tag{105}$$

A bit of algebra gives us:

$$\boxed{u/c = \frac{\lambda_{source}^2 - \lambda_{obs}^2}{\lambda_{source}^2 + \lambda_{obs}^2}} \tag{106}$$

which is useful to remember. If we can substitute in our known source wavelength ($\lambda_{source} = 479.8\,\text{nm}$) and our wavelength observed here on Earth ($\lambda_{obs} = 537.4\,\text{nm}$), we find that the star is moving with a velocity of

$$u = -0.11c \tag{107}$$

Notice that the observed wavelength was longer than the source wavelength. Longer wavelengths are towards the red end of the visible spectrum, and so we call this redshift. Redshifted light implies that the source is moving away from us. Hence we had a negative value for u (remember that we derived this equation for a source moving towards the observer with velocity u). Therefore the star is moving away at a velocity of $0.11c$.

6 Relativistic Velocity

In this and the following lectures we will discuss further implications of special relativity on momentum and energy. Since both momentum and kinetic energy depend on velocity, it's perhaps not surprising that our definitions for these will need to be modified. However, first we must study the relativistic equivalent of the *velocity* transformations which we looked at in Lecture 1.

6.1 Relativistic Velocity Transformations

Just as we replaced the Galilean position transformations with the Lorentz transformations, we must now replace the velocity transformations with their relativistic equivalents. A little care is needed here because there is one critical difference between position and velocity. For the position transformations, co-ordinates in which the reference frames aren't moving (y and z) experience no relativistic correction (i.e. $y = y'$ and $z = z'$). This isn't the case for velocity - the y and z velocities, v_y and v_z do experience a correction. This is because velocity is a function of time, and since the time co-ordinate *does* needs a relativistic correction, it follows that *all* velocities will also experience a correction, regardless of the direction.

The relativistic velocity transformations between frames S and S', where S is moving with velocity u along the x-axis with respect to S' are:

$$v_x = \frac{v'_x + u}{1 + uv'_x/c^2}, \qquad v_y = \frac{v'_y}{\gamma(1 + uv'_x/c^2)}, \qquad v_z = \frac{v'_z}{\gamma(1 + uv'_x/c^2)} \tag{108}$$

Similarly, we have the inverse transforms

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2}, \qquad v'_y = \frac{v_y}{\gamma(1 - uv_x/c^2)}, \qquad v'_z = \frac{v_z}{\gamma(1 - uv_x/c^2)} \tag{109}$$

As for the Lorentz transformation, it should be the case that these reduce to our familiar Galilean velocity transformation for speeds far below the speed of light ($u << c$). This is indeed true, because uv'_x/c^2 is then approximately equal to 0 and $\gamma \approx 1$ and so $v_x \approx v'_x + u$, $v_y \approx v'_y$ and $v_z \approx v'_z$.

It must also be the case that a velocity of c is one reference frame transforms into a velocity of c into another, since this is one of the postulates on which the Lorentz transformations are based. We can confirm this for light moving along the x-axis by substituting in $v'_x = c$, so that:

$$v_x = \frac{c + u}{1 + uc/c^2} \tag{110}$$

$$= \frac{c + u}{1 + u/c} \tag{111}$$

$$= \frac{c(c + u)}{c + u} \tag{112}$$

$$= c \tag{113}$$

For light moving along the y-axis, we can show the same thing. If we simply substitute $v_y = c$ into the equation for v'_y we get $v'_y = c/\gamma$. However, we have to remember that there will be both an x and y component to the transformed velocity (you can verify this by substituting $v_x = 0$ into the equation for v'_x, it can be seen that $v_x = u$).

The total velocity is then $\sqrt{v_x^2 + v_y^2}$, which can be shown to be c. This is an exercise on the problem sheet.

Also note the following:

- The equations for v_y and v_z contain v'_x in the denominator. That is, the transformation of the y and z velocities depends on the x velocity.
- Don't confuse the vs with u. u is the velocity of the frame of reference S' relative to S (along the x direction), while the vs and v's are the velocity of some object as measured in frames of reference S and S', respectively.

6.2 Derivation of the Relativistic Velocity Transformations

The relativistic velocity transformations can be derived from the Lorentz transformations. To make this simple, we first need to recognise that the Lorentz transformations are equally valid if we replace the x, y, z and t co-ordinates and their primed counterparts with intervals (or differences), Δx, Δy, Δz and Δt. We can make these intervals infinitesimally small, calling them, dx, dy, dz, and dt. Then we also note that the x, y and z components of velocity can be written $v_x = dx/dt$, and so on. So, starting with the velocity in the x-direction, we have the following to work with:

$$dx = \gamma(dx' + u dt'), \qquad dt = \gamma\left(dt' + u\frac{dx'}{c^2}\right), \tag{114}$$

and

$$v_x = \frac{dx}{dt}, \qquad v'_x = \frac{dx'}{dt'}. \tag{115}$$

Now, plug in the expressions for dx and dt into the equation for v_x, giving us

$$\begin{aligned} v_x &= \frac{dx}{dt} && (116)\\ &= \frac{\gamma(dx' + u dt')}{\gamma(dt' + u dx'/c^2)} && (117) \end{aligned}$$

The γ factors cancel,

$$v_x = \frac{dx' + u dt'}{dt' + u dx'/c^2}. \tag{118}$$

We want to end up with an equation containing velocities, v'_x, so let's divide through every term by dt'

$$v_x = \frac{dx'/dt' + u}{1 + udx'/c^2 dt'} \tag{119}$$

Now, we can simply substitute in $v'_x = dx'/dt'$ to get the equation for v_x in terms of v'_x.

$$\boxed{v_x = \frac{v'_x + u}{1 + uv'_x/c^2}.} \tag{120}$$

To derive the velocity in the y direction, v_y, we similarly start with

$$dy = dy', \qquad dt = \gamma(dt' + u\frac{dx'}{c^2}), \tag{121}$$

and

$$v_y = \frac{dy}{dt}, \qquad v'_y = \frac{dy'}{dt'}. \tag{122}$$

We perform a similar substitution to get

$$v_y = \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + udx'/c^2)}. \tag{123}$$

Notice it looks a bit different to the v_x case, we no longer have a γ on the top, because $dy = dy'$. Again, we divide through by dt' to get velocities:

$$v_y = \frac{dy'/dt'}{\gamma(1 + udx'/c^2 dt)}, \tag{124}$$

and so

$$\boxed{v_y = \frac{v'_y}{\gamma(1 + uv'_x/c^2)}.} \tag{125}$$

The derivation for v_z is the same, replacing z every time you see a y.

6.3 Relativistic Velocity Example

Relative Velocity of Two Rockets Example: *Two rockets are heading straight towards each other, both moving at $c/2$ relative to the Earth. What does one rocket measure for the velocity of the other?*

Solution: Common sense (and the Galilean velocity transformations) would tell us c. But we know that's not right. Instead, let's use the velocity transformation:

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} \tag{126}$$

We need to decide what v_x and u are. The velocities of the rockets are measured to be $c/2$ in the Earth's reference frame (S). So, for example, one of the rockets (call it rocket A) has $v_x = -c/2$ in the Earth's frame. Now we want to transform this velocity to the frame of rocket B (S'), so that we know what velocity rocket B measures rocket A to have. The reference frame of Rocket B is moving relative to Earth at $u = c/2$, and so:

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} \tag{127}$$

$$v_x = \frac{-c/2 - c/2}{1 - (-c/2)(c/2)/c^2} \tag{128}$$

$$v_x = \frac{-c}{1 + 1/4} \tag{129}$$

$$v_x = -0.8c \tag{130}$$

Which, as we would expect, is less than c.

7 Relativistic Momentum

In this lecture we will discuss further implications of special relativity on momentum. Since momentum is dependent on velocity, it's not surprising that our non-relativistic definition will need to be modified.

7.1 Non-Relativistic Momentum

In Newtonian mechanics, momentum is defined as:

$$p = mv \tag{131}$$

At first glance, there is no obvious reason why the thing that we call momentum should be defined in this way. Why not define momentum to be $p = m/v$ or $p = mv^8$? The reason that momentum is defined as mv is because this value is *conserved*. More generally, the total value of the momentum, summed over all objects, is conserved for any system that has no external forces acting on it.

To take a simple example, consider two blocks, each with a mass of 1 kg. One is stationary, and the other is moving towards it at a velocity of 10 m/s, as shown in the diagram in Figure 9.

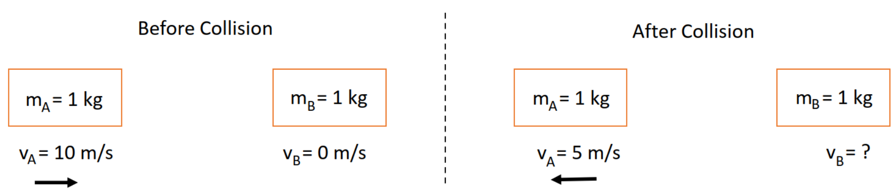

Figure 9: Before and after a collision between two blocks. The left diagram shows the two blocks moving together, A moving to the right with speed 10 m/s and B stationary. The right diagram shows the blocks after the collision, with A moving to the left with speed 5 m/s and the speed of B unknown.

The total momentum of the system before the two blocks collide is given by:

$$\begin{aligned} p &= m_1 v_1 + m_2 v_2 && (132)\\ &= (1\ \text{kg})(10\ \text{ms}^{-1}) + (1\ \text{kg})(0) && (133)\\ &= 10\ \ \text{kgms}^{-1} && (134) \end{aligned}$$

After the collision, the total momentum will still be equal to 10 kgm/s. So if, for example, the left-most block bounces back at 5 m/s, the second block must have a velocity v_2 such that:

$$\begin{aligned} p = 10\ \text{kgms}^{-1} &= (1\ \text{kg})(-5\ \text{ms}^{-1}) + (1\ \text{kg})v_2 && (135)\\ v_2 &= 15\ \text{ms}^{-1} && (136) \end{aligned}$$

Using the Galilean transformation between inertial frames, if momentum is conserved in one frame of reference then it will also be conserved in another. This doesn't mean that we calculate the same value of momentum for different reference frames. If an object is moving at 2 m/s in frame S and 4 m/s in S', then since $p = mv$, the value for momentum will be different in S and S'. But within S, a certain value of momentum, p, will be conserved (i.e. not change), and within S' a different value of p will be conserved.

Given that momentum conservation is such as well-established principle, we might hope that it continues to be conserved after we make relativistic transformations between reference frames. Unfortunately, in the way we have defined momentum, after we make velocity transformations between reference frames, momentum conservation will no longer be true for at least one of the frames. Below, we will show that this is the case, and then make a new definition of momentum, that reduces to the classical definition for $u << c$, but which is also correctly conserved in relativistic frames.

7.2 Breakdown of Momentum Conservation (Not Examinable)

To see that momentum as defined classically is not conserved in relativistic situations, consider the situation shown in Figure 7.2. There are two balls, ball A and ball B which are going to collide. We will consider two reference frames, S and S'. S' is moving with respect to S with a velocity u along the x-axis.

In reference frame S, A is moving directly up the y-axis with velocity $v_{A,y}$. In reference frame S', B moving directly down the y-axis (i.e. it has zero x-velocity) with velocity $v_{B,y} = -v_{A,y}$.

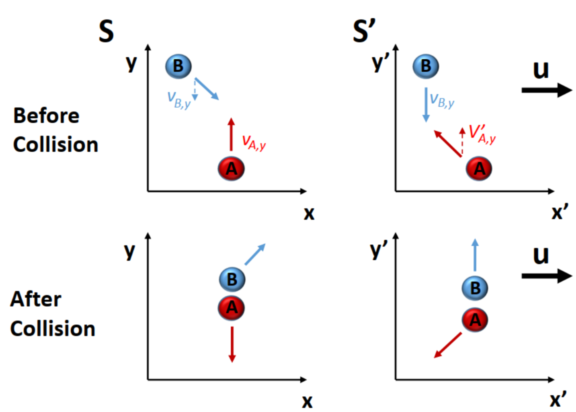

Figure 10: Collision between two balls, A and B, viewed from two different frames of reference, S and S'.

Let's consider what happens in reference frame S. We know the y-velocity of A, but we don't know the y-velocity of B in this frame, only in S'. In classical mechanics, $v_y = v'_y$, and so ball B simply has the opposite velocity and momentum to ball A. After a totally elastic collision, they would each then have the opposite velocity and momentum in the y direction.

In the relativistic case, however, we must use the relativistic velocity transform

$$v_y = \frac{v'_y}{\gamma(1 + uv'_x/c^2)}. \tag{137}$$

Ball B has no x-velocity in S', so $v'_x = 0$, and so we simply have:

$$v_y = \frac{v'_y}{\gamma} \tag{138}$$

Or, to put it another way, the y-velocity of B is smaller than that for A, and so the total momentum in the y-direction is not zero (it is positive because A, which is moving in the +y direction has more positive momentum than has B has negative momentum). But after the collision, the y-components of the velocity are reversed. Now there is more momentum in the -y direction, and so momentum is not conserved. Therefore, the classic definition of momentum is only approximately conserved for $u << c$. We need to work out a relativistic correction to momentum to recover its more general conservation.

7.3 Relativistic Momentum

The relativistic momentum of an object, measured in a reference frame in which the object is moving at velocity v, is given by:

$$p = \gamma m_0 v \tag{139}$$

Gamma is defined almost the same as before:

$$\gamma = \sqrt{\frac{1}{1 - v^2/c^2}} \tag{140}$$

except that we are using v rather than the velocity of the reference frame, u.

We have also introduced a new concept, the rest mass, m_0.

The **rest mass** of an object is its mass measured in its rest frame (the frame of reference in which it is not moving).

This implies that mass of an object is effectively different in different reference frames. We could think about defining this relativistic mass as something like $m = \gamma m_0$, but we tend not to write it this way unless we are writing the full equation for relativistic momentum. As ever, for low velocities, we have $\gamma \approx 1$ and so we recover the non-relativistic equation for momentum.

7.4 An Argument for Relativistic Momentum

We can motivate $p = \gamma m_0 v$ in the following way. Since $v = dx/dt$ we can write

$$p = m_0 v = m_0 \frac{dx}{dt} \tag{141}$$

When we change reference frame, then now $dt' \neq dt$. However, the proper time, $d\tau$, (the time measured in the object's reference frame) doesn't change with the reference frame. So, let's define:

$$p = m_0 \frac{dx}{d\tau}. \tag{142}$$

This is fine because in the rest frame of the object ($\gamma = 1$), we have $dt = d\tau$ by definition, so this is in agreement with non-relativistic physics in the limit $v \ll c$. More generally, for any reference frame, $dt = \gamma d\tau$, so we straight-away see that

$$p = m_0 \gamma \frac{dx}{dt} = \gamma m_0 v. \tag{143}$$

This isn't a true derivation of course, since we chose $p = m_0(dx/d\tau)$ without any real justification, but it shows that this is a plausible definition for relativistic momentum

8 Relativistic Energy

In this final lecture we will discuss one more relativistic correction - to energy. As we will see, while this correction is not quite as straightforward as the one for momentum, it has hugely important consequences, leading us to discover the equivalence between mass and energy.

8.1 Non-Relativistic Kinetic Energy

In non-relativistic Newtonian mechanics, the kinetic energy of an object is defined as

$$E_K = \frac{1}{2}mv^2, \tag{144}$$

where m is the object's mass and v is its velocity. As with momentum, this is the definition that we choose for kinetic energy precisely because it is this quantity which is conserved (providing we take into account all other types of energy in the system). Since v is frame-dependent, we expect E_K to also be frame dependent.

8.2 Relativistic Kinetic Energy

Based on the pattern we have seen so far, we might hope that relativistic energy might be as simple as $E_K = (1/2)\gamma m_0 v^2$. However, this is not the case. Instead, it is something very strange looking indeed.

$$\boxed{E_K = (\gamma - 1)m_0c^2. \tag{145}}$$

Firstly, we need not worry that there is no v in there, for this is hidden inside γ.

Before we work out where this equation comes from, let's check our requirement that this reduces to the familiar form for $v \ll c$. It is not a simple as saying that $\gamma = 1$ because then we have $E_K = 0$ which is what we want for $v = 0$, but doesn't tell us anything about small but non-zero values of v.

To get there, we instead make use of the binomial expansion. For small values of x, i.e $|x| \ll 1$,

$$(1-x)^{-1/2} = \frac{1}{\sqrt{1-x}} \approx 1 + x/2. \tag{146}$$

If we expand out γ in the equation for relativistic kinetic energy, we have

$$E_K = \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right)m_0c^2. \tag{147}$$

Then, using the binomial expansion, we obtain

$$E_k = (1 + v^2/2c^2 - 1)m_0c^2 \tag{148}$$

$$E_k = (1/2)m_0v^2, \tag{149}$$

which is exactly the non-relativistic expression for kinetic energy. So once again, it turns out that our familiar definition is really an approximation which is good only for $v \ll c$.

8.3 Derivation of Relativistic Kinetic Energy (Not Examinable)

To derive the equation for relativistic energy we need to recall that the change in kinetic energy of an object is given by the integral of the force applied over the distance it is applied through, i.e.

$$E_K = \int F ds, \tag{150}$$

where F is the force and s is the distance. Secondly, we recall that force is equal to the rate of change in momentum:

$$F = \frac{dp}{dt}. \tag{151}$$

Therefore, if we consider an object that has accelerated from rest to a velocity v, the kinetic energy is given by:

$$E_K = \int_{v=0}^{v=v} \frac{dp}{dt} ds. \tag{152}$$

Now, ds/dt is simply velocity, v, so

$$E_K = \int_{v=0}^{v=v} v \; dp, \tag{153}$$

and relativistic momentum is given by $p = \gamma m_0 u$, so that:

$$E_K = \int_{v=0}^{v=v} v d\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) = \int_{v=0}^{v=v} v \; d\left[(m_0 v)\left(1 - \frac{v^2}{c^2}\right)^{-1/2}\right], \tag{154}$$

where we have expanded out γ.

We want to get this integral in terms of dv so that we can integrate and apply the limits. To do this, we can use the chain rule to show that

$$\frac{d}{dv}\left[(m_0 v)\left(1 - \frac{v^2}{c^2}\right)^{-1/2}\right] = m_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2} + m_0\frac{v^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}. \tag{155}$$

If we divide the first term top and bottom by $(1 - v^2/c^2)$ to put everything over a common denominator we get:

$$d\left[(m_0 v)\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right] = m_0\left(1-\frac{v^2}{c^2}\right)^{-3/2} dv. \tag{156}$$

Substitute this back into the equation for E_K and we get

$$E_K = \int m_0\left(1-\frac{v^2}{c^2}\right)^{-3/2} v\ dv. \tag{157}$$

This integral can be solved using the substitution $w = \sqrt{1/(1-v^2/c^2)}$, to give

$$E_K = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} - m_0 c^2, \tag{158}$$

and so

$$\boxed{E_K = (\gamma - 1) m_0 c^2} \tag{159}$$

which is the expression for relativistic kinetic energy.

8.4 Total Energy

If we expand out our newly discovered expression for kinetic energy

$$E_K = \gamma m_0 c^2 - m_0 c^2, \tag{160}$$

we notice that it consists of one quantity subtracted from another. We might wonder what these two terms represent physically. As we will show below, it turns out that this equation really means:

$$\text{kinetic energy} = \text{total energy - rest energy} \tag{161}$$

and so,

$$\boxed{E_{total} = \gamma m_0 c^2} \tag{162}$$

8.5 Rest Energy and Mass-Energy Equivalence

In the rest frame of the object, where $v = 0$, then $\gamma = 1$ and so

$$E = m_0 c^2 \tag{163}$$

This shows that a non-moving object, i.e. an object viewed from its rest frame, still has some energy. We call this the **rest energy**.

We see that this rest energy depends only on the mass, m_0, and a constant, c. Since mass and energy are linked by a fixed constant, this demonstrates an equivalence between the two. In particle accelerators we can see this in action, as kinetic energy is converted to rest energy, and hence rest mass. So if we smash two particles together with a certain total rest mass, m_0, it doesn't follow that the rest mass of a new particle created must be equal to m_0.

8.6 Energy and Mass Units

The SI Unit for energy is the Joule, J. However, the kinds of objects we most often observe moving at speeds which are an appreciable fraction of c (and hence where the effects of special relativity become noticeable) tend to be particles (for example in a particle accelerator). In this case, the Joule is not a sensible unit to use. Instead we use electron-volts (eV).

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$$

(This comes from the energy gained by an electron moving across a voltage of 1 V, but that isn't important here.)

Since we know that rest mass is equivalent to rest energy, linked by the constant c^2, $E_0 = m_0 c^2$, we can also specify the rest mass in eV type units.

A particle with a rest energy of 1 eV would have a rest mass of 1 eV/c^2.

In practice we will actually find ourselves working with larger units of energy and mass, in which case we will talk about MeV and MeV/c^2, or GeV and GeV/c^2

We can also use these kinds of units for momentum. Recall that $p = \gamma m_0 v$. If we specify mass in MeV/c^2 and velocity as a fraction of c, then we can specify our momentum in units of MeV/c.

Using these kinds of units makes dealing with mass/energy/momentum problems in particle physics and relativity much simpler (since the cs also cancel out) and **you should be very comfortable with them going into the exam.**

8.7 Example Energy and Momentum Calculations

A neutron has a rest mass of 940 MeV/c^2 *and is travelling at 0.9c. What is its rest energy, total energy, kinetic energy and momentum?*

Solution:

- If the rest mass is 940 MeV/c^2, then to get the rest energy we use $E = mc^2 = (940\ \mathrm{MeV/c^2})c^2 =$ 940 MeV.

- The total energy is given by $E = \gamma mc^2$. For $v = 0.9c$ then $\gamma = \sqrt{\frac{1}{1-0.9^2}} = 2.294$.

- Therefore $E = (2.294)(940\ \mathrm{MeV}/c^2)c^2 = 2156$ MeV.

- Then, $E_K = (\gamma - 1)mc^2 = E - mc^2 = 2156$ MeV $- 940$ MeV $= 1316$ MeV.

- Finally, $p = \gamma mv = (2.294)(940\ \mathrm{MeV}/c^2)(0.9c) = 1940$ MeV/c.

- Note here that MeV units have been used throughout, making the problem much simpler.

  See also examples R-4 and R-5 in Tipler.

8.8 Demonstration that Massless Particles Travel at the Speed of Light

It is sometimes useful to write energy in terms of momentum. In classical mechanics, since $E_K = (1/2)mv^2$ and $p = mv$, we have

$$E = \frac{p^2}{2m} \tag{164}$$

The relativistic equivalent is:

$$E^2 = p^2c^2 + m_0^2c^4 \tag{165}$$

We can see this by recognising that $p = \gamma m_0 v$ and so $p^2 = \gamma^2 m_0^2 v^2$. So we can write

$$c^2p^2 = \gamma^2 m_0^2 c^4 \frac{u^2}{c^2}. \tag{166}$$

From $\gamma^2 = 1/(1 - u^2/c^2)$, a bit of algebra shows us that

$$c^2p^2 = \gamma^2 m_0^2 c^4 - m_0^2 c^4. \tag{167}$$

Since $E = \gamma m_0 c^2$, the first term is simply E^2, giving us

$$\boxed{E^2 = p^2c^2 + m_0^2c^4} \tag{168}$$

This is an important equation which relates the total energy, rest energy and momentum of a particle (without explicitly including the velocity). When the energy is much greater than the rest energy, it becomes $E \approx pc$.

Continuing, from $E = \gamma m_0 c^2$ and p $= \gamma m_0 v$

$$p = \frac{Ev}{c^2} \tag{169}$$

Now, if we consider a massless particle, so that $m_0 = 0$, then $E^2 = p^2c^2$ and we have the exact equation

$$E = pc. \tag{170}$$

Substituting in $p = Ev/c^2$ gives us

$$E = \frac{Evc}{c^2} \tag{171}$$

which immediately tells us that

$$v = c. \tag{172}$$

This shows that massless particles must have a velocity equal to c, the speed of light!

If we look more carefully, and remember $m_0 = 0$, then we see we had a division by zero when we calculated $p = Ev/c^2$ (and if $v = c$ then γ is undefined!). So this 'derivation', as we have presented it, is not entirely convincing. However, it turns out that $p = Ev/c^2$ *is* true when $m_0 = 0$ and $v = c$, in which case the only possible reconciliation with $E = pc$ is that $v = c$. More on this will have to wait until you learn about the energy-momentum 4-vector!