

Electromagnetism Part 5: Laplace and Poisson Equations, Uniqueness Theorem and Method of Images

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1 Introduction

Poisson's and Laplace's equations tell us something about how the potential behaves in regions of space. Laplace's equation is a special case of Poisson's equation, and applies to regions of space where there is no electric charge.

2 Derivation of Poisson and Laplace Equations

We have previously explored Gauss' Law,

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad (1)$$

and we also have an expression for $\vec{E}$ in terms of the potential, V ,

$$\vec{E} = -\nabla V. \quad (2)$$

Substituting the second equation into the first,

$$\nabla \cdot (-\nabla V) = \frac{\rho}{\epsilon_0}. \quad (3)$$

On the left we have the divergence of a gradient. This is the Laplacian operator; it acts on a scalar and gives another scalar. We normally write it as $-\nabla^2 V$, i.e.,

$$-\nabla^2 V = \frac{\rho}{\epsilon_0}, \quad (4)$$

or more usually written,

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}. \quad (5)$$

This is **Poisson's Equation**. In Cartesian co-ordinates it is,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon_0}. \quad (6)$$

Essentially, this tells us that the second derivative of the potential at any point in space depends on the charge density at that point.

For the special case of regions with no charge, i.e. $\rho = 0$, this is then known as **Laplace's Equation**,

$$\boxed{\nabla^2 V = 0}. \quad (7)$$

This tells us that at points in space where there is no charge, the second derivative of the potential must be zero. Of course, this 'special case' is actually quite common, unless we are inside a charged material, or sat exactly on a point charge, then $\rho = 0$.

Recall that the second derivative of a function is negative at a local maximum and positive at a local minimum. So, if the second derivative is zero at points where there is no charge, then the potential cannot be a maximum or a minimum at those points.

Note that Laplace's Equation also applies to many situations outside of electrostatics, for example fluid flow, but we will not study those here.

3 Solutions to Laplace's Equation in 1D

We are often interested in finding the potential outside of the charge distribution, i.e. where $\rho = 0$, so we would like to be able to solve Laplace's Equation. This is quite difficult in general, but it is interesting to study the form some simple solutions take.

We can begin by considering what would happen if the potential varied only along the x -direction. For example, let's consider the potential between two charged plates (separated along the x -axis).

In this case, since we know from symmetry that the potential only varies in the direction perpendicular to the plates, we would have a 1D form of Laplace's Equation,

$$\frac{d^2V}{dx^2} = 0. \quad (8)$$

This is a second order differential equation. By integrating twice we get a general solution of the form,

$$V(x) = C_1x + C_2, \quad (9)$$

where C_1 and C_2 are the integration constants, which we can determine if we know the boundary conditions, i.e. the value of V at specific points. For example, if we know $V(x = 0) = V_0$ and $V(x = a) = V_a$, then,

$$V(0) = C_1(0) + C_2 = V_0 \implies C_2 = V_0. \quad (10)$$

and,

$$V(a) = C_1(a) + C_2 = V_a \implies C_1 = \frac{V_a - V_0}{a}. \quad (11)$$

So that,

$$V(x) = V_0 + \frac{V_a - V_0}{a}x. \quad (12)$$

Notice that this equation describes a linear gradient of potential from $x = 0$ to $x = a$, as shown in Figure 1. This result has a very general significance. The solution to Laplace's Equation always gives us the 'smoothest' function between the boundaries. (Recall that we cannot have a maxima or minima because the second derivative is zero everywhere where there is no charge density).

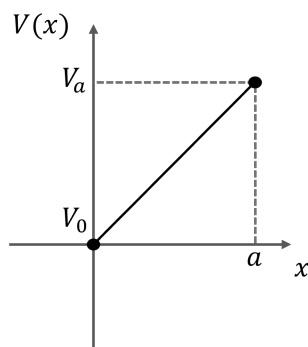


Figure 1: 1D Laplace example.

We can also obtain the x -component of the E-field from.

$$E_x = -\frac{dV}{dx} = -\frac{V_a - V_0}{a} \quad (13)$$

which, as expected, is a constant.

4 Solutions to Laplace's Equation in 3D

Let's take the simple example of a single point charge q , giving us a nicely symmetric problem. We first need to rewrite Laplace's Equation in spherical polar co-ordinates, which for a situation with spherical symmetry is simply,

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dV}{dr} \right) = 0, \quad (14)$$

where r is the radial co-ordinate and we are only considering $r \neq 0$. Multiplying both sides by r^2 and then integrating twice w.r.t. r , gives,

$$V(r) = -\frac{C_1}{r} + C_2, \quad (15)$$

The constant C_2 is the potential as $r \rightarrow \infty$, which we normally set to 0, so we have,

$$V(r) = -\frac{C_1}{r}. \quad (16)$$

This is indeed what we expect. We would need to know one more boundary condition to obtain the C_1 constant, which we know should be $-\frac{q}{4\pi\epsilon_0}$. Or, alternatively, we could obtain the radial component of the field,

$$E_r = -\frac{dV}{dr} = -\frac{C_1}{r^2}, \quad (17)$$

and then from Gauss's Law remembering that the radial field is parallel to the surface vector everywhere,

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \quad (18)$$

$$\oint E_r dA = \frac{q}{\epsilon_0} \quad (19)$$

$$4\pi r^2 E_r = \frac{q}{\epsilon_0} \quad (20)$$

$$E_r = \frac{q}{4\pi\epsilon_0 r^2} \quad (21)$$

from which we can immediately read-off the correct value of C_1 .

5 Dirichlet and Neumann Boundary Conditions

For Laplace's equation, the solution is uniquely determined by the boundary values of V . This is called **Dirichlet boundary conditions**. That is to say, if we know the values of the potential along the boundary of a region, then we automatically know the values of the potential everywhere inside that boundary.

The solution is also determined uniquely (up to an additive constant) by the normal derivative of the potential at the boundary, i.e. the rate of change of the potential along the normal to the boundary,

$$\frac{\partial V}{\partial n} = -E_n, \quad (22)$$

which we see is just the component of the electric field in the normal direction. So, if we know the electric field at the boundary, we also uniquely fix the potential everywhere inside the boundary.

The normal component of the electric field at a point outside but very close to the boundary is given by,

$$E_n = \frac{\sigma}{\epsilon_0}. \quad (23)$$

(We saw that this was true for a plane of charge, so if it helps just think of each little segment of the boundary as being a small plane.)

So the normal component only depends on the surface charge density, σ , and so the potential inside the surface only depends on σ . If we know the surface charge density everywhere along a boundary (which may be zero in a lot of places) then again we know the form of the potential inside the surface, up to the addition of a constant factor (and since our choice of zero for potential is just a convention, e.g. we set zero potential at infinity, this makes no physical difference.)

6 Uniqueness Theorem and Method of Images

When we apply Dirichlet or Neumann boundary conditions, we obtain a solution for the potential inside the boundary, and this solution is **unique**, i.e. there can be no other solution (up to an additive constant for Neumann). A simple example of boundary conditions in practice would be a conductor that we set to a fixed voltage by connecting it to a power supply.

We can use this uniqueness in the **method of images**. This is essentially a trick that helps us to solve electrostatics problems involving conductors that would otherwise be difficult to solve.

The idea is to imagine replacing the conductor with a set of charges, called image charges, which give the same boundary conditions but which allows us to calculate the potential more easily. Since we know the same boundary conditions give the same potential, we have then solved the problem.

Of course the challenge is to choose the image charges correctly. The canonical example is as follows.

Example: We have a plane conductor, which is grounded, so that $V = 0$ on the conductor. A charge q is placed a perpendicular distance a from the plane conductor, as shown in figure 2(a). We would like to work out the electric field and potential in the region $z > 0$.

You should first think about how you could solve this in any other way. Remember that the charge on the conductor is not zero, there must be some non-zero charge to cancel out the effect of the positive charge to make the potential zero along the conductor. So it is not trivial!

To solve this using the method of images, we replace the plane conductor with a charge of $-q$ a distance a on the other side of where the plane conductor was. Let us say the plane conductor is on the $z = 0$ plane, $(x, y, z) = (x, y, 0)$, and the real charge q is at $(0, 0, a)$. We then put the image charge at $(0, 0, -a)$.

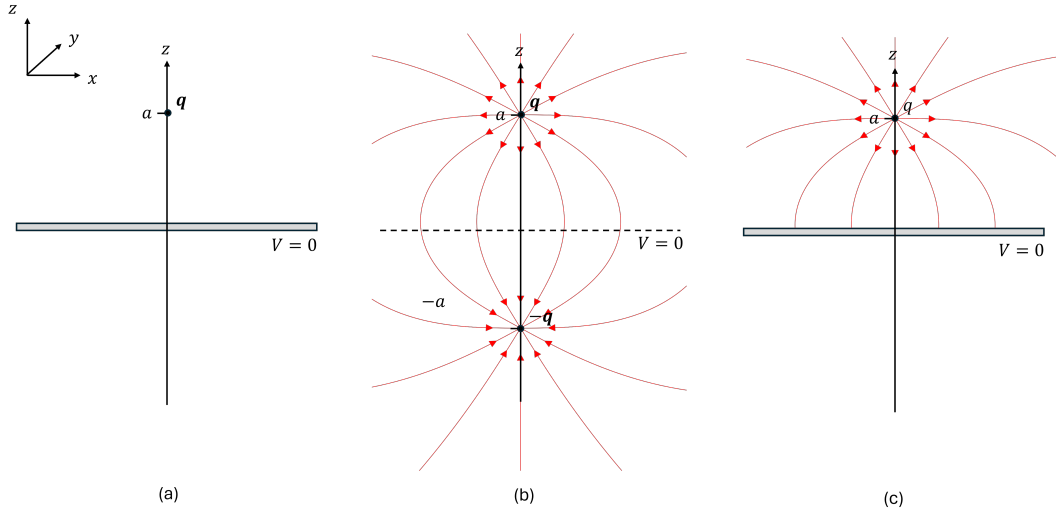


Figure 2: (a) Plane conductor fixed to $V = 0$ and charge q a distance a away; (b) equivalent system using point charges by method of images, also showing electric field lines; (c) electric field lines for original system (only valid for $z > 0$).

By inspection, we clearly still have $V = 0$ on the $z = 0$ plane due to the real and image charges. So this is valid for the method of images, we haven't changed the boundary conditions for the region $z > 0$.

Therefore, for the region $z > 0$, the potential due to these charges is,

$$V(r) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{\sqrt{x^2 + y^2 + (z - a)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z + a)^2}} \right), \quad z > 0.$$

This is true both for our method of images setup, using the two charges, but also the original problem with a plane conductor and a charge - the potential is the same!

The electric field is $\vec{E} = -\nabla V$, the field lines are plotted in Figure 2(b,c).

In order to maintain $V = 0$ on the conductor in the presence of charge q there must be some surface charge induced on the surface. Using $\sigma = \epsilon_0 E_z$ we find this is,

$$\sigma(\rho) = -\frac{q a}{2\pi(\rho^2 + a^2)^{3/2}}, \quad \rho = \sqrt{x^2 + y^2}.$$

The force on the point charge (attraction toward the plane) equals the Coulomb force between q and its image:

$$\vec{F} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2a)^2} \hat{k} = -\frac{q^2}{16\pi\epsilon_0 a^2} \hat{k}.$$

So, using the method of images, we can solve for V and E everywhere where $z > 0$, and have also found the force on the charge.

Note that the method **cannot be used for arbitrary situations**, it is only helpful when we can replace the charge distribution with a convenient set of point charges while keeping the boundary

conditions on the potential the same. These situations will always involve a symmetry of some kind.

For the purposes of the exam, other than for the very simple example above, I would always tell you what the equivalent charges are - no need for 'inspired guesses'.