

Electromagnetism Part 2: Electric Field

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1 Charge and Coulomb's Law

Electric charges may be positive or negative. Charges experience forces of attraction (between unlike sign charges, e.g. positive and negative) or repulsion (between like charges, e.g. positive and positive). This Coulomb or electrostatic force is given by,

$$\boxed{\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_{12}|^2} \hat{r}_{12}.} \quad (1)$$

This is the force on charge q_2 that is a displacement $\vec{r}_{12}$ from charge q_1 . $\hat{r}_{12}$ is the unit vector pointing from charge q_1 to charge q_2 i.e.,

$$\hat{r}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|}, \quad (2)$$

where $|\vec{r}|$ is the length of the vector pointing from the charge q_1 to charge q_2 , i.e. simply the distance between the charges. This is illustrated in Figure 1.

Note that the Coulomb force is an inverse square law, i.e. it varies with $1/r^2$. From Newton III, the force on charge q_1 is $\vec{F}_{21} = -\vec{F}_{12}$, which follows simply because $\hat{r}_{21} = -\hat{r}_{12}$.

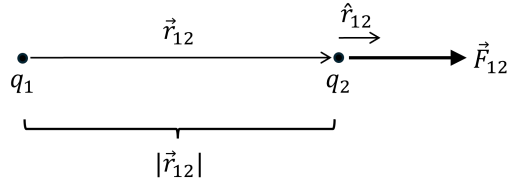


Figure 1: Illustration of Coulomb force.

For some numerical calculations it is useful to remember that,

$$\frac{1}{4\pi\epsilon_0} \approx 9 \times 10^9, \quad (3)$$

and this is sometimes labelled k or Coulomb's constant.

2 Important Note on Vector Notation

When we have a vector, such as $\vec{E}$ or $\vec{r}$, the complete way to write its magnitude (i.e. its length) is as $|\vec{E}|$ or $|\vec{r}|$. This makes it clear that we do have a vector, but are only referring to its magnitude (a scalar). However, it gets tedious to write/typeset this, so I will mostly use E and r for the magnitude instead.

3 Electric Field

If we have a point charge q , then the electric field vector, $\vec{E}$, a displacement $\vec{r}$ away is,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}|^2} \hat{r}, \quad (4)$$

where $\hat{r}$ is the unit vector pointing from the charge to the point in the field, i.e.,

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|}. \quad (5)$$

The scalar $|\vec{r}|$ is the length of the vector pointing from the charge to the point in the field, i.e. it is simply the distance of the point from the charge.

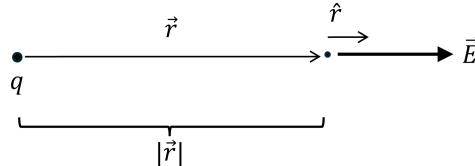


Figure 2: Illustration of electric field calculation from a point charge.

The units of electric field are **Newtons per Coulomb**, **N/C**.

By comparing the equations for the Coulomb force and the electric field, we can see immediately that if we place a charge of q_0 in an electric field then it will experience a Coulomb force of,

$$\boxed{\vec{F} = q_0 \vec{E}}, \quad (6)$$

and, if it has a mass m , an acceleration of,

$$\vec{a} = \frac{\vec{F}}{m} = \frac{q_0 \vec{E}}{m}. \quad (7)$$

4 Superposition Principle

If we have multiple charges then the total electric field is simply the vector sum of the electric fields created by each charge, $\vec{E}_i$. If we have N charges then,

$$\vec{E} = \sum_{i=1}^N \vec{E}_i = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^N \frac{q_i}{|\vec{r}_i|^2} \hat{r}_i, \quad (8)$$

where $\vec{r}_i$ is the vector from the i th charge to the point where we are calculating the field, and $\hat{r}_i$ is the corresponding unit vector.

4.1 Electric field superposition example

Consider two charges. Charge 1, $q_1 = 4 \text{ mC}$, lies on the y-axis at $y = 4 \text{ m}$ and charge 2, $q_2 = 2 \text{ mC}$, lies at the point $x = 3 \text{ m}$, $y = 4 \text{ m}$. What is the electric field at the origin ($x = 0$, $y = 0$)?

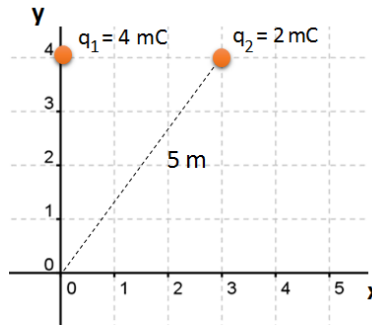


Figure 3: Layout of charges in superposition example.

For the electric field due to charge 1, we have $\vec{r}_{10} = -4\hat{j} \text{ m}$, and so $r_{10} = 4 \text{ m}$ and $\hat{r}_{10} = -\hat{j}$. Then,

$$\begin{aligned}
\vec{E}_1 &= \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_{10}^2} \hat{r}_{10} \\
&= (9 \times 10^9 \text{ Nm}^2\text{C}^{-2}) \frac{(4 \times 10^{-3} \text{ C})}{(4 \text{ m})^2} (-\hat{j}) \\
&= -2.25 \times 10^6 \hat{j} \text{ N/C}
\end{aligned}$$

For the electric field due to charge 2, we have $\vec{r}_{20} = -3\hat{i} - 4\hat{j}$ m. The magnitude is therefore $r_{20} = \sqrt{3^2 + 4^2} = 5$ m and the unit vector,

$$\hat{r}_{20} = \frac{\vec{r}_{20}}{r_{20}} = \frac{-3\hat{i} - 4\hat{j}}{5}$$

Then:

$$\begin{aligned}
\vec{E}_2 &= \frac{1}{4\pi\epsilon_0} \frac{q_2}{r_{20}^2} \hat{r}_{20} \\
&= (9 \times 10^9 \text{ Nm}^2\text{C}^{-2}) \frac{(2 \times 10^{-3} \text{ C})}{(5 \text{ m})^2} \left(\frac{-3\hat{i} - 4\hat{j}}{5} \right) \text{ N/C} \\
&= -(4.32 \times 10^5) \hat{i} - (5.76 \times 10^5) \hat{j} \text{ N/C}
\end{aligned}$$

The total field, by superposition, is then,

$$\begin{aligned}
\vec{E} &= \vec{E}_1 + \vec{E}_2 \\
&= -(4.32 \times 10^5) \hat{i} - (2.83 \times 10^6) \hat{j} \text{ N/C}
\end{aligned}$$

5 Continuous Distribution of Charges

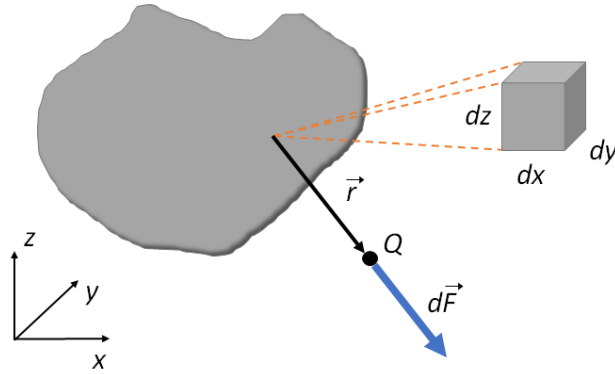
In many cases we will not be dealing with a few isolated charges, but a continuous distribution of charges. In which case, rather than summing the forces, we can perform an integral over the charge *density*.

The electric field due to a distribution of charges with charge density ρ is given by the volume integral

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho dV}{|\vec{r}|^2} \hat{r} \quad (9)$$

where $\vec{r}$ is the position vector from the volume element dV to the point where we calculate the field, with $\hat{r}$ the corresponding unit vector.

This is illustrated in Figure 5. Essentially this says we integrate over the field from each infinitesimal point in the volume of charge. Bear in mind that ρdV is *charge density* $\times$ *volume element*, which is



just the charge of the infinitesimal volume, dV , i.e. $dq = \rho dV$. We normally think of the element as a cube with sides dx , dy and dz .

To solve this for an arbitrary charge distribution we would have to use a computer, and approximate the distribution as a very large number of point charges (this is a simple example of something called *finite element modelling*). However, for certain types of symmetric distributions we can also find analytical solutions.

5.1 Electric field from a rod

We have a rod of length $2L$ with a uniform charge along it. We will define the linear charge density, λ . λ is simply the total charge divided by the length, i.e.

$$\lambda = \frac{q_{rod}}{2L} \quad (10)$$

We would like to know the electric field at position P at vertical position $\vec{R}$ from the centre of the rod, as shown in Figure 4. We will call the direction along the rod x , and the two end points are $x = -L$ and $x = L$.

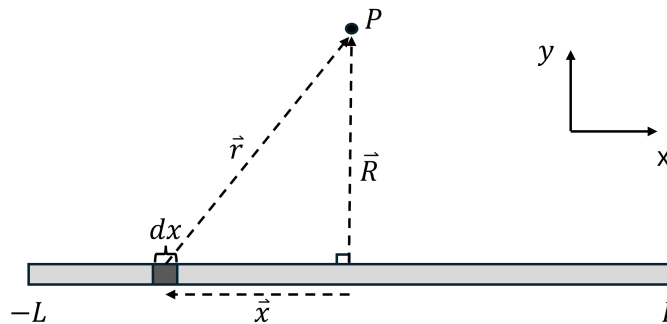


Figure 4: Geometry for line of charge calculation

Consider a small piece of the rod, of length dx . This has a charge $dq = \lambda dx$ (i.e. the length times the linear charge density). We can therefore replace ρdV in the definition above with $dq = \lambda dx$.

Hence the electric field due to this element is:

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{|\vec{r}|^2} \hat{r} \quad (11)$$

To obtain $\vec{E}$ we would like to integrate along the rod (i.e. dx), but notice that $\vec{r}$ depends on the x position of the element. We see that:

$$\vec{r} = \vec{R} - \vec{x} \quad (12)$$

Let us call the distance of the point from the rod z , i.e $\vec{R} = z\hat{j}$, so that

$$\vec{r} = z\hat{j} - x\hat{i}, \quad (13)$$

and then,

$$|\vec{r}|^2 = z^2 + x^2, \quad (14)$$

and,

$$\hat{r} = \frac{1}{\sqrt{z^2 + x^2}}(z\hat{j} - x\hat{i}). \quad (15)$$

Now we can proceed,

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{z^2 + x^2} \frac{1}{\sqrt{z^2 + x^2}}(z\hat{j} - x\hat{i}) \quad (16)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{(z^2 + x^2)^{3/2}}(z\hat{j} - x\hat{i}) \quad (17)$$

This is the field due to one element of the rod, so to obtain the total field from the rod we integrate over the length,

$$\vec{E} = \int_{-L}^L d\vec{E} \quad (18)$$

We can simplify here by noting the symmetry of the problem. There is as much of the rod to the left of the point as there is to the right. So the horizontal component of the field will be zero, i.e. we only keep the $\hat{j}$ component:

$$\vec{E} = \frac{\lambda z}{4\pi\epsilon_0} \int_{-L}^L \frac{dx}{(z^2 + x^2)^{3/2}} \hat{j} \quad (19)$$

This integral can be solved either by looking up the standard integral:

$$\int \frac{dx}{(z^2 + x^2)^{3/2}} = \frac{x}{z^2 \sqrt{z^2 + x^2}} + C \quad (20)$$

(I will give you this in the exam if you need it) or, if you would like to practice your maths, by the substitution $x = r \tan u$ and $dx = r \sec^2(u) du$ (have fun). The solution is:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\lambda}{z} \frac{x}{(z^2 + x^2)^{1/2}} \hat{j} \Bigg|_{x=-L}^{x=L} \quad (21)$$

Putting in the limits:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\lambda}{z} \left[\frac{L}{(z^2 + L^2)^{1/2}} - \frac{-L}{(z^2 + L^2)^{1/2}} \right] \hat{j} \quad (22)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q_{rod}}{z} \left[\frac{1}{(z^2 + L^2)^{1/2}} \right] \hat{j} \quad (23)$$

where we have substituted $q_{rod} = 2\lambda L$, since the total charge on the rod is the charge density multiplied by its length. Let's think about the extreme case, where we are a long way from the rod, i.e. $z \gg L$, in which case we have that the magnitude of the field is:

$$E \approx \frac{1}{4\pi\epsilon_0} \frac{q_{rod}}{z} \left[\frac{1}{(z^2)^{1/2}} \right] \hat{j} = \frac{1}{4\pi\epsilon_0} \frac{q_{rod}}{z^2} \quad (24)$$

which is the field from a point charge. A long way from a finite length rod, it looks like a point, so this is what we expect.

Now consider the case when the rod is very long compared to the distance of Q from the rod, i.e. $L \gg z$, in which case:

$$\frac{L}{(z^2 + L^2)^{1/2}} \approx \frac{L}{(L^2)^{1/2}} = 1 \quad (25)$$

and so

$$\boxed{E \approx \frac{1}{2\pi\epsilon_0} \frac{\lambda}{z}} \quad (26)$$

This is the general expression for the field a distance z from an infinitely long rod of charge density λ . For a rod of length L it is a good approximation providing $z \ll L$ and we don't look at the field anywhere too close to either end of the rod. **Notice that the force drops off as $1/z$, not $1/z^2$!**

5.2 Electric field from a disc of charge

Consider a disc of charge as shown below, with charge density (i.e, charge per unit area) of σ . What is the electric field a perpendicular distance z away from the centre, at point P ?

To tackle this we will think of the disc as being made up of rings, calculate the field due to a ring at a given radius, and then integrate over the radius of the disc, R .

For a ring at radial position ρ , each point on the ring is at the same distance $r = \sqrt{z^2 + \rho^2}$ from point P , with the unit vector,

$$\hat{r} = \frac{1}{(z^2 + \rho^2)^{1/2}} (z\hat{z} - \rho\hat{\rho}). \quad (27)$$

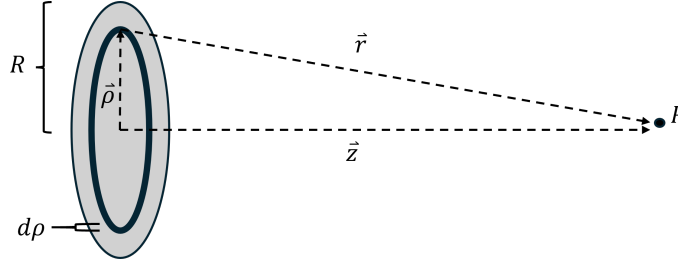


Figure 5: Geometry for disc of charge calculation

The ring has an area of $2\pi\rho d\rho$. (This is the circumference of the ring, $2\pi\rho$ multiplied by an infinitesimal thickness $d\rho$.) The charge on it is therefore the area multiplied by the charge density, i.e. $dq = 2\pi\sigma\rho d\rho$.

As before we have a symmetry, so that we expect to only have a field pointing along the z-direction. The z-component due to the ring at radius ρ is then

$$d\vec{E}_z = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \hat{r} \quad (28)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{2\pi\sigma\rho z d\rho}{(z^2 + \rho^2)^{3/2}} \hat{z} \quad (29)$$

$$= \frac{\sigma z}{2\epsilon_0} \frac{\rho d\rho}{(z^2 + \rho^2)^{3/2}} \hat{z} \quad (30)$$

Now we integrate over the radius of the disc:

$$\vec{E} = \frac{\sigma z}{2\epsilon_0} \int_0^R \frac{\rho d\rho}{(z^2 + \rho^2)^{3/2}} \hat{z} \quad (31)$$

By making the substitution $u = z^2 + \rho^2$, so that $du = 2\rho d\rho$ we obtain

$$\vec{E} = \frac{\sigma z}{2\epsilon_0} \left(-\frac{1}{\sqrt{\rho^2 + z^2}} \right) \hat{z} \Big|_{\rho=0}^{\rho=R} \quad (32)$$

Evaluating the limits gives

$$\vec{E} = \frac{\sigma z}{\epsilon_0} \left(-\left[\frac{1}{\sqrt{R^2 + z^2}} - \frac{1}{z} \right] \right) \hat{z} \quad (33)$$

$$\vec{E} = \frac{\sigma}{\epsilon_0} \left[1 - \frac{z}{\sqrt{R^2 + z^2}} \right] \hat{z} \quad (34)$$

5.3 Electric field on-axis from an infinite plane of charge

To solve this one we will go back to the result for a disc of charge of finite radius, and make this radius infinite.

$$\vec{E} = \frac{\sigma z}{2\epsilon_0} \frac{1}{\sqrt{\rho^2 + z^2}} \hat{z} \Big|_{\rho=0}^{\rho=\infty} \quad (35)$$

Which evaluates to

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{z} \quad (36)$$

Notice that there is no dependence on z . So the electric field is constant with distance from the infinite plane of charge. Of course this is for an infinite plane of charge, but for a plane of charge which is large compared to the distance z of the the point where we calculate the field this is a good approximation. So we have a constant and uniform electric field close to a large plane of charge, or between two planes of charge, such as in a parallel plate capacitor.