Vectors Review

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1 Introduction

In much of electromagnetism we will deal with electric and magnetic fields. These are vector fields, meaning that we specify a vector at every point in space which tells us the magnitude and direction of the field at that point.

You will therefore need to have a good understanding of vectors and be comfortable manipulating them. You have covered vectors already, and below is only a brief summary of some points which are most relevant.

2 Definition of a Vector

A vector has both magnitude and direction, in contrast to a scalar which has only magnitude.

We denote a vector quantity using bold, such as $\mathbf{E}$, an underscore, such as $\underline{\mathrm{E}}$, or an arrow on top, such as $\vec{E}$. The latter two can be used when writing by hand.

In electromagnetism we will deal with vectors in 3D space, although in some situations we only need to worry about two dimensions.

The magnitude of the vector we then write as $|\vec{E}|$ or simply E. For spatial vectors, i.e. the vector pointing from one point in space to another, the magnitude is the length of the vector. For other vectors, such as force, the magnitude is not a length, it is the strength or size of the force.

3 Components of a vector

The components of a vector are defined for a certain co-ordinate system.

In electromagnetism we will begin by using the familiar Cartesian co-ordinate system (x,y,z).

A vector $\vec{A}$ that points A_x units in the x-direction, A_y units in the y-direction and A_z units in the z-direction can be written in the form (A_x, A_y, A_z) or $A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$.

$\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are examples of **unit vectors**, vectors which have a length (magnitude) of 1. $\hat{\imath}$ points along the positive direction of the x-axis, $\hat{\jmath}$ along the y-axis, and $\hat{k}$ along the z-axis. Because any vector in 3D space can written in terms of these three unit vectors, we say that these vectors form a basis, and that they are **basis vectors**.

4 Vector Magnitude/Length

If we have a vector defined in Cartesian co-ordinates, $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$, the magnitude of the vector is given by:

$$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Example. *The vector* $10\hat{\imath}$ *has a magnitude of* $\sqrt{10^2} = 10$.

Example. *The vector* $3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ *has a magnitude of* $\sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$.

5 Position Vectors

A point in 3D space is sometimes represented by a position vector. This is the vector from the origin of the co-ordinate system (0,0,0) to the point.

Example. *A point* p *at* $x = 3$, $y = 2$, $z = -2$ *has a position vector* $\vec{p} = (3, 2, -2)$ *or* $\vec{p} = 3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$.

If point A is at (A_x, A_y, A_z) and point B is at (B_x, B_y, B_z) then the vector from A to B is:

$$\vec{AB} = (B_x - A_x)\hat{\imath} + (B_y - A_y)\hat{\jmath} + (B_z - B_z)\hat{k}$$

If $\vec{A}$ and $\vec{B}$ are the position vectors of point A and B (i.e. $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$) then the vector from A to B can also be written as:

$$\vec{AB} = \vec{B} - \vec{A} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k} - (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) = (B_x - A_x)\hat{\imath} + (B_y - A_y)\hat{\jmath} + (B_z - A_z)\hat{k}$$

which of course is exactly the same thing.

6 Unit Vectors

We will use unit vectors throughout to describe the directions of electric forces and fields.

Above we defined three specific unit vectors: $\hat{\imath}$,$\hat{\jmath}$, and $\hat{k}$, which form a basis. However, any vector which has a magnitude/length of 1 is also a unit vector. If we multiply a scalar by a unit vector we don't change the magnitude of the scalar, but we endow it with a direction.

If we have a vector $\vec{A}$, we can obtain the unit vector $\hat{A}$ pointing in the same direction by diving by the magnitude:

$$\hat{a} = \frac{\vec{A}}{|\vec{A}|}$$

Example. *What is the unit vector which points from point A (4,6) towards point B (12,10)?*

First write down the vector which points from A to B.

$$\vec{F} = (12 - 4)\hat{\imath} + (10 - 6)\hat{\jmath} = 8\hat{\imath} + 4\hat{\jmath}$$

To make this a unit vector we need it to have a magnitude of 1. So we determine its magnitude:

$$|\vec{F}| = \sqrt{8^2 + 4^2} = \sqrt{80}$$

Then divide:

$$\hat{F} = \frac{8\hat{\imath} + 4\hat{\jmath}}{\sqrt{80}}$$

$$\hat{F} = 0.90\hat{\imath} + 0.45\hat{\jmath}$$

7 Scalar (Dot) Product

The scalar or dot product between two vectors is given by:

$$\vec{A}.\vec{B} = |A||B|\cos\theta$$

where θ is the angle between the two vectors.

The dot product can be thought of as a 'projection' of one vector onto another. Loosely speaking, it tells us how much of vector $\vec{A}$ is pointing in the direction of vector $\vec{B}$.

If two vectors are orthogonal (perpendicular) then the dot product is zero. If two vectors are parallel then the dot product is the product of their magnitude/length.

This means that $\hat{\imath}.\hat{\imath} = 1$, $\hat{\jmath}.\hat{\jmath} = 1$, $\hat{k}.\hat{k} = 1$, while $\hat{\imath}.\hat{\jmath} = 0$, $\hat{\jmath}.\hat{k} = 0$ and $\hat{k}.\hat{\imath} = 0$. (Recall that the magnitude of these basis unit vectors is 1).

In Cartesian co-ordinates, if $\vec{A} = (A_x, A_y, A_z)$ and $\vec{B} = (B_x, B_y, B_z)$ then the dot product can be calculated by:

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

8 Vector (Cross) Product

The vector or cross product between two vectors is given by:

$$\vec{A} \times \vec{B} = |A||B|\sin\theta\ \hat{n}$$

where $\hat{n}$ is a unit vector that is perpendicular to both $\vec{A}$ and $\vec{B}$. This leaves two possible directions (i.e. $\hat{n}$ could point positive or negative), the equations below tell you the direction for Cartesian co-ordinates, and when we cover magnetic fields we will discuss simple ways to remember which way it points.

The cross product, loosely speaking, tells us the extent to which two vectors are perpendicular to each other. The unit vector points perpendicular to the plane in which the two original vectors lie.

Vectors which are parallel have a zero cross product, so we have $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{\jmath} \times \hat{\jmath} = 0$.

If we take cross products of non-identical unit vectors then we get the other unit vector in a cyclical manner, so that $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$.

However, the cross product does not commute (i.e. the order of the two vectors changes the direction of the cross product). So $\hat{\jmath} \times \hat{\imath} = -\hat{k}$, $\hat{k} \times \hat{\jmath} = -\hat{\imath}$, $\hat{\imath} \times \hat{k} = -\hat{\jmath}$.

In Cartesian co-ordinates, the cross product is calculated as:

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_z)\hat{k}$$

If you know the rules for calculating the determinant of a 3x3 matrix, it is easy to remember the cross product as:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \hat{\imath} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \hat{\jmath} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \hat{k}$$

You can practice using this to see if you can reproduce the rules for cross products of the unit vectors above.

Example. *What is the cross product of* $\vec{A} = 2\hat{\imath} + 4\hat{\jmath} + 1\hat{\jmath}$ *and* $\vec{B} = \hat{\imath} - 2\hat{\jmath} + 3\hat{\jmath}$*?*

$$\begin{aligned} \vec{A} \times \vec{B} &= (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k} \\ &= [(4)(3) - (1)(-2)]\hat{\imath} + [(1)(1) - (2)(3)]\hat{\jmath} + [(2)(-2) - (4)(1)]\hat{k} \\ &= 14\hat{\imath} - 5\hat{\jmath} - 8\hat{k} \end{aligned}$$