Electromagnetism Part 3: Electric Potential and Conductors

Mike Hughes

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1 The Curl of E

We saw in the previous section that,

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}.$$

What about $\nabla \times \vec{E}$?

The Curl of a vector field tells us how much it 'swirls' around a point. If we think of how the electric field from a charge looks like, it doesn't seem to do much swirling. And since the superposition principle tells us that the field from any charge distribution is just the sum of the fields, we might expect that in general the curl is zero.

To show this more formally, we can use **Stoke's Theorem** which states that for any vector field, $\vec{E}$, the curl of the field integrated over a surface is related to the line integral of the field around the perimeter of that surface:

$$\int_S (\nabla \times \vec{E}) \cdot d\vec{A} = \oint_{\partial S} \vec{E} \cdot d\vec{l}$$

We will use the simplest case of a point charge and try to compute the RHS of this equation (and hope that it comes out to be zero).

In general, the line integral of $\vec{E}$ between any two points a and b is

$$\int_a^b \vec{E} \cdot d\vec{l} = \int_a^b \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}\, dr$$

Notice that we have replaced $\vec{E} \cdot d\vec{l}$ with $E\, dr$ on the RHS. We can do this because we know the field from a point charge is radial, i.e. the line integral only accumulates when we are moving in a radial direction. Let's do the integral:

$$\int_a^b \vec{E} \cdot d\vec{l} = \frac{q}{4\pi\epsilon_0} \frac{-1}{r}\bigg|_a^b = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right)$$

Now we wanted to integrate over a closed loop, so $a = b$, in which case,

$$\oint \vec{E} \cdot d\vec{l} = 0$$

This is true for any closed loop, and so Stoke's theorem tells us then immediately that,

$$\nabla \times \vec{E} = 0.$$

We have shown this for a single point charge, and the superposition principle tells us this will be true for any collection of charges (or distribution of charges), and so the theorem holds.

(Actually, we should have a caveat here that this is only true for non-moving charges, which we have been implicitly assuming all the time - we are doing electo*statics*.)

2 Electric Potential

The result that $\nabla \times \vec{E} = 0$ has a very significant consequence. It means that the **electric field is a conservative field**. If we move a charge around in an electric field, the potential energy it gains or loses only depends on its initial and final positions - not the path it takes. And if we return it to its starting point, the change in energy is zero.

This is allows us to define a **scalar potential field**, V. At each point in space within an electric field we have a single value that tells us the potential energy a charge of 1 C would have if placed there.

We define the potential at a point $\vec{r}$ as:

$$V(\vec{r}) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}$$

i.e. it is the integral of the electric field from infinity to the point $\vec{r}$. This might seem a bit odd, but the electric field will generally drop off to zero at infinity so it actually makes things simpler.

More generally, we can express the difference in potential between two points, a and b, as,

$$V(b) - V(a) = -\int_{a}^{b} \vec{E} \cdot d\vec{l}$$

in which case we see that the re-obtain the previous equation by setting $V(a = \infty) = 0$.

The units of electrostatic potential are **Joules per Coulomb**, J/C, otherwise known as **Volts**, V; which we use depends on what seems most natural by the context; if we are thinking about energy gain as a charge moves through a potential we would tend to use J/C, while anything to do with electric circuits would always be in V.

2.1 Electric Potential from a Point Charge

The electric potential due to a point charge is,

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}|} \tag{1}$$

Note that we have been explicit here about specifying $|\vec{r}|$ to be is the magnitude of the distance, i.e. it is always positive. (We can write r as we did with the electric field from a point source, but just remember this is the **magnitude** of a vector, it cannot be negative).

We can show this using the definition of potential,

$$V(\vec{r}) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}$$

Field lines from a point source are radial, hence $\vec{E} \cdot d\vec{l} = E\,dr$,

$$\begin{aligned} V &= -\int_{\infty}^{r} E\,dr & (2)\\ &= -\frac{q}{4\pi\epsilon_0} \int \frac{1}{r^2}\,dr & (3)\\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{r}\bigg|_{\infty}^{r} & (4)\\ & & (5) \end{aligned}$$

which gives,

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}} \tag{6}$$

Note that potential falls off as $1/r$, while the field falls of as $1/r^2$. Similarly, the Coulomb force between two charges goes as $1/r^2$ while the potential energy of those two charges goes as $1/r$.

2.2 Electric Potential Energy

Don't confuse electric potential with electric potential energy. The potential energy is the energy a charge would have if placed in the field. Since the potential tells us the energy a charge of 1C would have, the energy U for a charge q is simply,

$$U = qV.$$

This has SI units of of Joules, as you would expect.

We can check this makes sense as follows. Consider a charge, q, which is generating an electric field, and a second test charge, q_0 which we will move. If an infinitesimal force $d\vec{F_W}$ is applied to move q_0 an infinitesimal distance $d\vec{dl}$, then the work done is

$$dW = d\vec{F_W} \cdot d\vec{l} \tag{7}$$

In moving q_0 from point A to point B, the work done is given by:

$$W = \int_A^B \vec{F_W} \cdot d\vec{l} \tag{8}$$

where $\vec{F}$ is the force applied. In order to move the charge without giving it a resultant velocity, the force applied would need to be equal and opposite to the electrostatic force, F, i.e.

$$\Delta U = -\int_A^B \vec{F} \cdot d\vec{l} \tag{9}$$

This tells us the difference in potential energy between two points. It is only ever really the difference in energy we care about, that is the only thing that has a physical meaning. If we want to specify the absolute potential energy, rather than a difference, we need to choose some reference point at which we will say the electric potential energy is zero (just like how me might choose the surface of the Earth as the point with zero gravitational potential energy). We are free to choose this wherever we like. By convention we choose this point as a point which is infinitely far away from the charge q. This turns out to be convenient because infinitely far away from a charge, the force and the electric field from that charge goes to zero.

The force, F, at a distance r is given by Coulomb's Law

$$\vec{F} = \frac{kqq_0}{r^2}\hat{r} \tag{10}$$

and so to find out the potential energy for q_0 at a distance r from this charge, we need to imagine bringing in q_0 from infinity (i.e. we calculate the difference in energy from this point with infinity). So we need to integrate from ∞ to r:

$$U = -\int_{\infty}^{r} \vec{F}.\vec{dl} = -\int_{\infty}^{r} \frac{kqq_0}{r^2}\hat{r}.\vec{dl} \tag{11}$$

Now, $\hat{r}.\vec{dl}$ (a dot product) is simply dr. So we have:

$$\begin{aligned} U &= -\int_{\infty}^{r} \frac{kqq_0}{r^2} dr && (12)\\ &= -\left[-\frac{kqq_0}{r}\bigg|_{\infty}^{r} \right] && (13)\\ &= k\frac{qq_0}{r} && (14)\\ &= q_0 V && (15) \end{aligned}$$

which is our definition of electric potential energy.

3 Electric Potential from Continuous Charge Distributions

Because electric potential obeys the super-position principle, we can also sum the potential due to a collection of discrete charges or integrate over a continuous charge distribution. This is actually easier than for the electric field because potential is a scalar rather than a vector.

3.1 Electric Potential Along the Axis From a Line of Charge

Consider a rod of uniform linear charge density λ and length L. What is the electric potential a distance d from the end of the rod, along the axis?

An element of length dx has charge $dq = \lambda\, dx$. The electric potential from this element, dV, is given by:

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{L-x+d} = \frac{1}{4\pi\epsilon_0}\frac{\lambda\, dx}{L-x+d} \tag{16}$$

where x is the position of the element along the rod and hence $(L-x)+d$ is the distance to the point where we are calculating the potential. The potential is therefore:

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\lambda \int_0^L \frac{dx}{L-x+d} && (17)\\ &= -\frac{1}{4\pi\epsilon_0}\lambda \ln(L-x+d)\bigg|_{x=0}^{L} && (18)\\ &= -\frac{1}{4\pi\epsilon_0}\lambda\Big[\ln(d) - \ln(L+d)\Big] && (19)\\ &= \frac{1}{4\pi\epsilon_0}\lambda \ln\left[1+\frac{L}{d}\right] && (20) \end{aligned}$$

As a check, when we are far from the rod, $l \ll d$, then:

$$V = \frac{1}{4\pi\epsilon_0}\lambda \ln[1 + L/d] \approx \frac{1}{4\pi\epsilon_0}\frac{\lambda l}{d} = \frac{1}{4\pi\epsilon_0}\frac{Q}{d} \tag{21}$$

at which point it looks like a point charge.

3.2 Electric Potential at a Perpendicular Distance From a Line of Charge

Consider a rod of uniform charge density λ and length L. What is the electric potential a distance z from the centre of rod, perpendicular to the rod?

As before, an element of length dx has charge $dq = \lambda dx$, giving rise to a potential of,

$$dV = \frac{1}{4\pi\epsilon_0}\frac{\lambda\, dx}{r}, \tag{22}$$

where r is the magnitude of the distance from the element to the point where we are calculating the potential. From geometry, $r = \sqrt{x^2 + z^2}$, and so, if we label the two ends of the rod as a and b, then the electric potential is given by:

$$\begin{aligned}
V &= \frac{1}{4\pi\epsilon_0}\lambda \int_{-L}^{L} \frac{dx}{r} && (23)\\
&= \frac{1}{4\pi\epsilon_0}\lambda \int_{-L}^{L} \frac{dx}{\sqrt{x^2 + z^2}} && (24)\\
&= \frac{1}{4\pi\epsilon_0}\lambda \ln(\sqrt{x^2 + z^2} + x)\Big|_{x=-L}^{x=L} && (25)\\
&= \frac{1}{4\pi\epsilon_0}\lambda \ln\left[\frac{\sqrt{L^2 + z^2} + L}{\sqrt{L^2 + z^2} - L}\right] && (26)
\end{aligned}$$

3.3 Electric Potential From a Shell of Charge

If we have a spherical shell of charge q and radius R, as shown in figure 1(a), then we know from Gauss's Law that the E-field outside the shell is the same as if all the charge was a point charge at the centre of the sphere, and the E-field inside the shell is zero (since there is no enclosed charge). We can then easily calculate the electric potential by integrating.

First outside the shell,

$$\begin{aligned}
V(r) &= -\int_{\infty}^{r} \vec{E} \cdot d\vec{l} && (27)\\
&= -\frac{q}{4\pi\epsilon_0}\int_{\infty}^{r} \frac{1}{r^2}\, dr && (28)\\
&= \frac{q}{4\pi\epsilon_0\, r} && (29)
\end{aligned}$$

This isn't a surprise, it's the potential from a point charge. Now, inside the shell, we have an integral of two parts, one the same as the integral above, and (from the surface of the shell, $r = R$ inwards), one in which $E = 0$,

$$V(r) = -\int_R^r \vec{E} \cdot d\vec{l} - \int_\infty^R \vec{E} \cdot d\vec{l} \tag{30}$$

$$= \frac{q}{4\pi\epsilon_0 R} \tag{31}$$

We now have an expression for V that doesn't depend on r, so V is constant, but **not zero** inside the shell. In fact it has the same value as the potential right on the surface of the shell, where $r = R$. This is illustrated in Figure 1(b) below, where it is plotted along with the electric field.

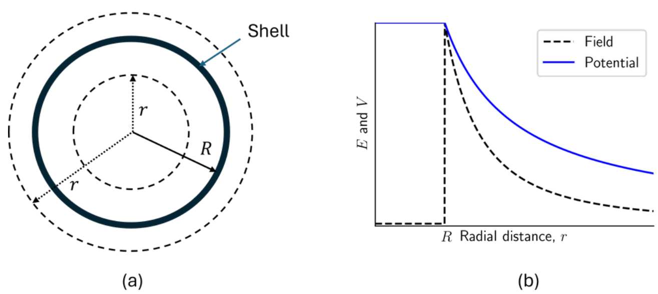

Figure 1: (a) Spherical shell, showing Gaussian surfaces (dashed lines) inside and outside shell; (b) plot of E and V as a function of distance from centre of shell.

4 Obtaining Electric Potential from Electric Field

Lets explicitly calculate the electric potential for some continuous charge contributions using the equation we wrote earlier,

$$V = -\int_\infty^r \vec{E}.d\vec{l} \tag{32}$$

First, we will calculate the potential at a point P, a perpendicular distance r from an infinite line of charge of linear charge density λ. In this idealised case of an infinite rod we cannot put the reference potential at zero as we will find the integral diverges. This is only a problem because we have infinite rod, for any finite length rod the field eventually starts to drop off as $1/r^2$ when we get far enough away.

To solve the infinite rod, we will calculate the potential relative to a known potential at distance r_0, $V(r_0)$.

We have previously calculated the electric field using Gauss's Law to be:

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}\hat{r} \tag{33}$$

We will integrate along a field line from r_0 to point P at r. This means that $\vec{E}.\vec{dl} = Edr$,

$$V(r) - V(r_0) = -\int_{r_0}^{r} E\,dl \tag{34}$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\int_{r_0}^{r}\frac{1}{r}\,dr \tag{35}$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\ln(r)\Big|_{r_0}^{r} \tag{36}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_0}{r}\right) \tag{37}$$

Similarly for an infinite plane of charge of surface charge density σ, where we have,

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{r} \tag{38}$$

the field at infinity is obviously not going to be zero, so we again calculate the potential relative to the potential at distance r_0,

$$V(r) - V(r_0) = -\int_{r_0}^{r} E\,dl \tag{39}$$

$$= -\frac{\sigma}{2\epsilon_0}\int_{r_0}^{r} dr \tag{40}$$

$$= -\frac{\sigma}{2\epsilon_0}r\Big|_{r=r_0}^{r} \tag{41}$$

$$= \frac{\sigma}{2\epsilon_0}(r_0 - r) \tag{42}$$

5 Obtaining Electric Field from Electric Potential

The electric field E is related to the potential V by,

$$\vec{E} = -\nabla V.$$

The gradient of a scalar gives us a vector, for example in Cartesian co-ordinates,

$$\vec{E} = \nabla V = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}.$$

The idea that we can obtain a vector field from a scalar field might seems quite strange, as though we end up with more information than we started with. But bear in mind that, in electrostatics, $\nabla \times \vec{E} = 0$ which significantly constrains the possibilities of what $\vec{E}$ can be.

6 Lines of Equipotential and Electric Field Lines

The gradient of a scalar field is a vector field that at each position in space points in the direction of the greatest increase in the scalar field. Since $\vec{E} = -\nabla V$, the electric field points in the direction of the greatest decrease in potential.

We can sketch the scalar field as a series of lines of equipotential, lines which join points of equal potential (similar to how contour lines join points of equal height on a map). The electric field lines will then always point perpendicular to the line of equipotential at each point. Examples of this are shown in Figure 2.

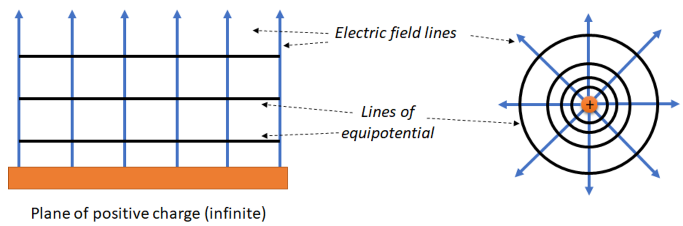

Figure 2: Lines of equipotential and field lines for a plane of charge (left) and a point charge (right)

7 Potential and Field in Conductors

We can think more about conductors. A conductor contains free charge carriers (in metals these are electrons) that are able to move throughout the material. If an electric field exists inside a conductor, these charges will experience a force and will move.

7.1 Electric Field Inside a Conductor

Suppose we place a conductor in an external electric field. The free charges will immediately begin to move under the influence of the field. They continue to move until they redistribute themselves in such a way that the net electric field inside the conductor is zero.

If this were not the case, the charges would still feel a force and would continue to move. Therefore, in **electrostatic equilibrium**,

$$\vec{E} = 0 \quad \text{inside a conductor.} \tag{43}$$

This has several very important consequences.

7.2 Potential is Constant Inside a Conductor

Since we have shown that

$$\vec{E} = -\nabla V,$$

if $\vec{E} = 0$ inside a conductor, then

$$\nabla V = 0.$$

This means that the potential **does not change with position inside the conductor**. Therefore,

$$V = \text{constant inside a conductor.} \tag{44}$$

Not only is the potential constant throughout the volume of the conductor, it is also constant over its surface. The entire conductor is therefore an **equipotential**.

7.3 Charge Resides on the Surface

We can choose to draw a Gaussian surface entirely within a conducting material, and we are free to do this however we like. Regardless of the shape of the surface, $\vec{E} = 0$ everywhere on the surface, and so Gauss's Law tells us,

$$\oint \vec{E} \cdot d\vec{A} = 0 = \frac{Q_{\text{enc}}}{\epsilon_0}.$$

Therefore,

$$Q_{\text{enc}} = 0.$$

This tells us that there can be no net charge within the bulk of the conductor in electrostatic equilibrium. **Any excess charge must reside on the surface**.

7.4 Electric Field Just Outside a Conductor

Although the electric field inside a conductor is zero, just outside the surface it is generally non-zero. In fact, the electric field at the surface must be perpendicular to the surface.

If there were a tangential component of the electric field at the surface, charges would experience a force parallel to the surface and would move. This would contradict electrostatic equilibrium. Therefore, the electric field at the surface must be precisely normal to the surface.

Using Gauss's Law with a small "pillbox" Gaussian surface that straddles the conductor surface, we can show that the magnitude of the electric field just outside the surface is

$$E = \frac{\sigma}{\epsilon_0}, \tag{45}$$

where σ is the surface charge density at that point.

7.5 Cavities and Shielding

If a conductor contains a hollow cavity and no charge is placed inside that cavity, then the electric field inside the cavity is also zero. The conductor shields the interior from external electric fields. This effect is known as **electrostatic shielding**.

If a charge is placed inside a cavity, charges will be induced on the inner surface of the cavity so that the electric field within the conducting material itself remains zero. The net induced charge on the inner surface will be equal and opposite to the charge placed in the cavity.

This principle is the basis of a **Faraday cage**, which protects sensitive equipment from external electric fields by surrounding it with a conducting enclosure.

7.6 Charge Conservation and Charging By Induction

Electric charge cannot be created or destroyed; it can only be transferred from one object to another. The total charge of an isolated system always remains constant.

This principle helps us understand what happens when we charge a conductor by **induction**, as shown in Fig. 3.

Consider a neutral conducting sphere. Its total charge is zero, meaning it contains equal amounts of positive and negative charge.

- Bring a positively charged rod close to (but not touching) the sphere. The free electrons in the conductor are attracted toward the rod, while positive charge is left on the far side. The sphere is still overall neutral, the charges have merely redistributed.
- Now bring in another sphere, touching the first. Charges flow from the new sphere to the original.
- Separate the spheres and each retains its charge, even when we remove the rod.

Or, alternatively,

- Take a single sphere next to the rod and connect the far side of the sphere to ground. Because the rod is attracting electrons, electrons flow from the Earth onto the sphere.
- Remove the ground connection while the rod is still present. The extra electrons remain on the sphere.
- Finally remove the rod. The electrons redistribute themselves uniformly over the surface. But we now have a net negative charge.

7.7 Summary of Conductor Properties

In electrostatic equilibrium:

- $\vec{E} = 0$ inside a conductor.
- The potential V is constant throughout the conductor.
- Excess charge resides on the surface.
- The electric field just outside the surface is perpendicular to the surface and has magnitude $E = \sigma/\epsilon_0$.

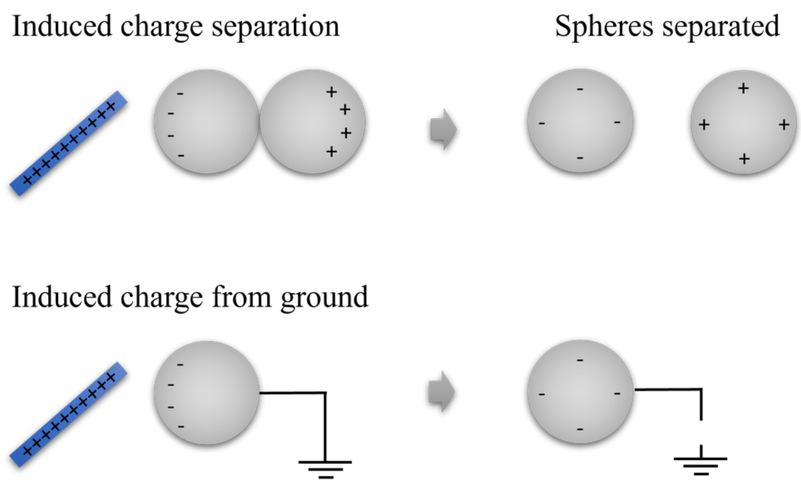

Figure 3: Two examples of charging by induction.

- Conductors provide electrostatic shielding.