Electromagnetism Part 8: Magnetic Field

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1 Introduction

In this section we introduce magnetic forces, their effect on moving charges and currents, and the fundamental equations governing magnetic fields.

2 Electric Current

First, a reminder about electric current. Consider when we have charges flowing along a cylinder (such as a wire). The cross-section of the cylinder has area S, the charges have charge q and move at speed v, and there are n charges per unit volume.

The electric current is the charge moving through a cross section of the wire in time interval ΔT,

$$I = \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

The charge passing through a cross section in unit time is $qnvS\Delta t$ and so,

$$I = \frac{qnvS\Delta t}{\Delta t} = qSnv$$

Current has units of Amperes, $\mathrm{A} = 1\,\mathrm{C/s}$.

The current density, J, is the current per unit of cross section area, i.e.,

$$J = \frac{I}{S} = qnv$$

Current density has units of $\mathrm{A/m^2}$.

3 Lorentz Force

If we have a magnetic flux density, $\vec{B}$, then a charge q moving at velocity $\vec{v}$ in that field experiences a force, $\vec{F}$ given by:

$$\vec{F} = q\vec{v} \times \vec{B}$$

This defines the unit of magnetic field, the Tesla (T). Note that the magnetic flux density is sometimes known informally as the magnetic field strength (as in Griffiths), although strictly this is $H = B/\mu_0$, where μ_0 is the permeability of free space, $4\pi \times 10^{-7}\,\mathrm{T\,m/A}$.

$$1\mathrm{T} = 1\frac{\mathrm{N/C}}{\mathrm{m/s}} = 1\frac{\mathrm{N}}{\mathrm{A\,m}}. \tag{1}$$

In practice we rarely experience fields anywhere near 1 T; the magnetic field from the Earth is around 10^{-4} T, an amount which is also known as 1 Gauss (G). The magnets in modern Magnetic Resonance Imaging (MRI) systems can reach a few Tesla, and require strict protections to prevent magnetic materials entering the controlled area and being pulled in the magnet.

Note that the force is the cross product of the velocity and the field. So it is perpendicular to both, i.e. $\vec{F}.\vec{B} = 0$. A convenient way to remember the direction in which the force points on a positive charge is the **Right Hand Rule**, shown in Figure 1. Curl your fingers as if rotating the velocity vector into the magnetic field vector. Your thumb then points in the direction of the force.

The total force on a charge depends on any electric field present, i.e.,

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

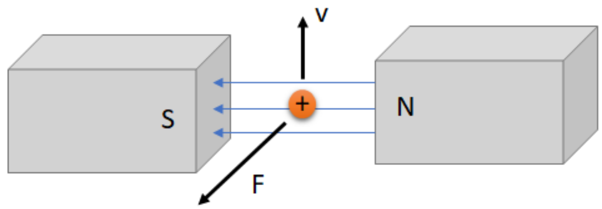

Figure 1: Rule for direction of Lorentz force.

3.1 Work done by a Magnetic Field

Recall that the work done by a force is

$$W = \int \vec{F} \cdot d\vec{r} \tag{2}$$

where $d\vec{r}$ is the vector describing the motion of the particle. But the magnetic force is always perpendicular to the motion of the particle, i.e. $\vec{F}.\vec{dr} = 0$. So the magnetic force does no work. This means it cannot change the kinetic energy of a particle. Or, in other words, it changes the direction of the velocity, but not the magnitude (the speed).

3.2 Lorentz Force on a Current Carrying Wire

Now think back to the current flowing through a wire.

Treat this as a line of charge with charge density λ, as shown in Figure 2 so that a line of length dl has charge $dq = \lambda dl$. If the line is moving at speed v (what we call the drift velocity of the charges) this gives current $I = \lambda v$.

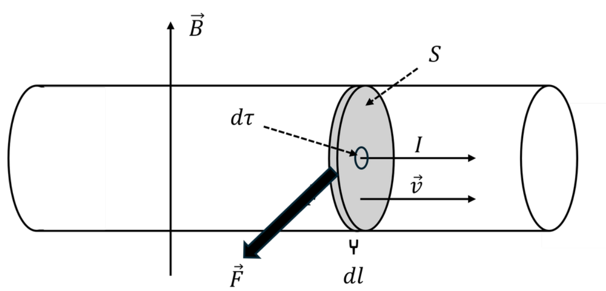

Figure 2: Force on a current carrying wire.

We define a vector for the current density such that, $JS\,d\vec{l} = \vec{J}\,d\tau = \vec{v}\,dq$, where $d\tau$ is a volume element, i.e. so that we have,

$$\vec{v}dq = Id\vec{l} = \vec{J}d\tau$$

If the wire is placed in magnetic field $\vec{B}$, then the Lorentz force on the differential length of wire $d\vec{l}$ is,

$$d\vec{F} = dq\,(\vec{v} \times \vec{B}) = I\,d\vec{l} \times \vec{B}.$$

If we have a straight wire, $\vec{L}$, in a uniform field then we can easily calculate the total force,

$$\vec{F} = I\vec{L} \times \vec{B},$$

with magnitude,

$$F = BIL\sin(\theta),$$

where L is the length of the wire and θ is the angle between the wire and the field.

In terms of the current density $\vec{J}$, the force on a volume element $d\tau$ is

$$d\vec{F} = \vec{J} \times \vec{B}\,d\tau.$$

4 Continuity Equation

One of the fundamental principles of physics is the conservation of electric charge. Charge cannot be created or destroyed, only moved from one place to another.

Consider a fixed volume V enclosed by a closed surface S. The total charge inside the volume is

$$Q = \int_V \rho\,d\tau, \tag{3}$$

where $d\tau$ is a volume element.

The rate of change of charge inside the volume is therefore

$$\frac{dQ}{dt} = \frac{d}{dt}\int_V \rho\,d\tau = \int_V \frac{\partial \rho}{\partial t}\,d\tau. \tag{4}$$

Charge can only change within the volume if current flows through the boundary surface. The net current flowing out of the volume is given by the surface integral

$$\oint_S \vec{J} \cdot d\vec{S}, \tag{5}$$

where $d\vec{S}$ is an outward-pointing surface element.

Conservation of charge requires that the decrease of charge inside the volume equals the net outward current,

$$\frac{dQ}{dt} = -\oint_S \vec{J} \cdot d\vec{S}. \tag{6}$$

Combining these expressions gives the integral form of the continuity equation,

$$\boxed{\int_V \frac{\partial \rho}{\partial t}\, d\tau + \oint_S \vec{J} \cdot d\vec{S} = 0.} \tag{7}$$

Using the divergence theorem,

$$\oint_S \vec{J} \cdot d\vec{S} = \int_V \nabla \cdot \vec{J}\, d\tau, \tag{8}$$

the integral form can be converted into the differential form,

$$\boxed{\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0.} \tag{9}$$

where $\vec{J}$ is the current density and ρ is the charge density.

To understand this equation, consider a small volume of space. The term $\nabla \cdot \vec{J}$ represents the net flow of charge out of the volume per unit time. If more charge leaves the volume than enters it, the charge density inside the volume must decrease. This change is described by the term $\partial\rho/\partial t$.

Rearranging the continuity equation,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}, \tag{10}$$

shows explicitly that a decrease in charge density corresponds to a positive divergence of the current density.

In the special case of steady currents, the charge density does not change with time, so

$$\frac{\partial \rho}{\partial t} = 0, \tag{11}$$

and the continuity equation reduces to

$$\nabla \cdot \vec{J} = 0. \tag{12}$$

Therefore, in steady-state conditions, charge does not accumulate at any point in the circuit.

5 Biot-Savart Law

We now consider how electric currents produce magnetic fields. The magnetic field generated by a steady current distribution is given by the **Biot–Savart law**.

Consider a small element of current $I\, d\vec{l}$ at position $\vec{r}'$, and let $\vec{r}$ be the position at which we wish to calculate the magnetic field. The vector pointing from the current element to the field point is

$$\vec{R} = \vec{r} - \vec{r}', \tag{13}$$

with magnitude $R = |\vec{R}|$.

The contribution $d\vec{B}$ to the magnetic field at $\vec{r}$ from the current element is

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \vec{R}}{R^3}. \tag{14}$$

The total magnetic field is obtained by integrating over the entire current-carrying path,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{I\, d\vec{l} \times \vec{R}}{R^3}. \tag{15}$$

The direction of the magnetic field is determined by the cross product $d\vec{l} \times \vec{R}$ and can be found using the right-hand rule. Curling the fingers of the right hand in the direction of the current, the thumb points in the direction of the magnetic field lines.

The Biot–Savart law applies to steady currents and is the magnetic analogue of Coulomb's law in electrostatics. While Coulomb's law describes the electric field produced by static charges, the Biot–Savart law describes the magnetic field produced by moving charges.

5.1 Magnetic Field of a Straight Current-Carrying Wire

As an important example, consider an infinitely long straight wire carrying a steady current I. By symmetry, the magnetic field forms concentric circles around the wire, with magnitude depending only on the distance r from the wire.

Using the Biot-Savart law, the magnetic field at distance r from the wire can be shown to be,

$$\boxed{B = \frac{\mu_0 I}{2\pi r}.}$$

This is an exercise on this week's problem sheet.

The direction of the field is given by the right-hand rule: the thumb points along the current, and the curled fingers indicate the direction of the magnetic field.

This result is consistent with Ampere's law, which provides an alternative and often simpler method for calculating magnetic fields in highly symmetric situations.

6 Ampere's Law

In many situations the magnetic field produced by a current distribution can be found directly using the Biot–Savart law. However, when the current distribution has a high degree of symmetry, a simpler method is available.

Ampere's law relates the circulation of the magnetic field around a closed path to the total current passing through the surface bounded by that path.

The integral form of Ampere's law in free space is,

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}},$$

where the line integral is taken around a closed loop C, and I_{enc} is the total current passing through any surface bounded by C.

The direction of the line element $d\vec{l}$ is chosen according to the right-hand rule: curling the fingers of the right hand in the direction of integration, the thumb points in the direction of the enclosed current.

Ampere's law as written is **valid only for steady currents**, for which the charge density does not change with time. In time-dependent situations, the law must be modified, as will be discussed later in the module.

6.1 Example: Magnetic Field of a Long Straight Wire

As an example, consider an infinitely long straight wire carrying a steady current I. By symmetry, the magnetic field forms concentric circles around the wire, and its magnitude depends only on the distance r from the wire.

Choose an Amperian loop to be a circle of radius r centred on the wire. Along this loop, the magnetic field has constant magnitude and is everywhere parallel to $d\vec{l}$.

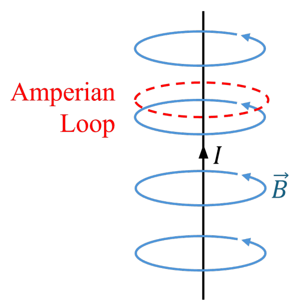

Figure 3: Ampere's Law applied to a long straight wire.

Ampere's law then gives

$$\oint_C \vec{B} \cdot d\vec{l} = B \oint_C dl = B(2\pi r).$$

The enclosed current is simply I, so Ampere's law becomes

$$B(2\pi r) = \mu_0 I,$$

and hence

$$B = \frac{\mu_0 I}{2\pi r},$$

in agreement with the result obtained from the Biot-Savart law.

6.2 Differential Form of Ampere's Law

The integral form of Ampere's law can be converted into a local, differential form using Stokes' theorem,

$$\oint_C \vec{B} \cdot d\vec{l} = \int_S (\nabla \times \vec{B}) \cdot d\vec{S}.$$

Since the surface S is arbitrary, we have,

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J},}$$

again, valid (in this form) **only for steady currents**.

6.3 Magnetic Field of a Long Solenoid

A solenoid is a long coil of wire consisting of many closely spaced turns, carrying a steady current I. Solenoids are widely used to generate approximately uniform magnetic fields.

Consider a solenoid of length L with N turns, so that the number of turns per unit length is

$$n = \frac{N}{L}.$$

We assume the solenoid is long compared to its radius, so that edge effects can be neglected. In this case, the magnetic field inside the solenoid is uniform and directed along its axis, while the magnetic field outside is very small.

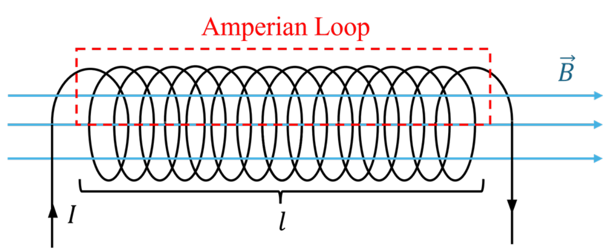

Figure 4: Ampere's Law applied to a solenoid.

To find the magnetic field, we apply Ampere's law,

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}.$$

Choose a rectangular Amperian loop, with one side of length l inside the solenoid and parallel to its axis, and the opposite side outside the solenoid. The contributions from the sides perpendicular to the field vanish, since $\vec{B} \cdot d\vec{l} = 0$ there. The contribution from the side outside the solenoid can be neglected because the magnetic field is approximately zero.

The line integral therefore reduces to

$$\oint_C \vec{B} \cdot d\vec{l} = Bl.$$

The current enclosed by the loop is equal to the number of turns within length l multiplied by the current in each turn,

$$I_{\text{enc}} = nlI.$$

Ampere's law then gives,

$$Bl = \mu_0 n l I,$$

and hence the magnetic field inside the solenoid is

$$B = \mu_0 n I.$$

The magnetic field inside a long solenoid is uniform and depends only on the current and the number of turns per unit length. The direction of the field is given by the right-hand rule: curling the fingers of the right hand in the direction of the current, the thumb points along the direction of the magnetic field inside the solenoid.

7 Gauss's Law for Magnetism

Finally, we have an equivalent of Gauss's Law for magnetism. The integral form of Gauss' law for magnetism states that the total magnetic flux through any closed surface is zero,

$$\boxed{\oint_S \vec{B} \cdot d\vec{S} = 0.}$$

This result holds for all magnetic fields, regardless of their source. It implies that magnetic field lines never begin or end, but instead form continuous closed loops.

The absence of any net magnetic flux through a closed surface means that there are **no isolated magnetic charges, or magnetic monopoles**. While magnetic monopoles are of theoretical interest, none have been observed experimentally.

Using the divergence theorem,

$$\oint_S \vec{B} \cdot d\vec{S} = \int_V (\nabla \cdot \vec{B})\, d\tau,$$

the integral form can be converted into a differential, or local, form. Since the volume V is arbitrary, Gauss's law for magnetism becomes,

$$\boxed{\nabla \cdot \vec{B} = 0.}$$