

Electromagnetism Part 3: Gauss's Law

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1 Flux

The flux through a surface can be thought of as the number of electric field lines that pass through that surface. More rigorously, it is the scalar (dot) product between the electric field vector and the surface normal, integrated over the whole surface,

$$\phi_E = \int_S \vec{E} \cdot \hat{n} dA, \quad (1)$$

where dA is the area of an infinitesimal element of the surface and $\hat{n}$ is a vector that is perpendicular to the surface at each point, known as the surface normal. Instead of $\hat{n}dA$ we often write $d\vec{A}$ instead, so you can think of $d\vec{A}$ as a vector that is perpendicular to the surface and equal in magnitude to dA ,

$$\phi_E = \int_S \vec{E} \cdot d\vec{A}. \quad (2)$$

The scalar product between two vectors is zero if they are perpendicular, i.e. there are no field lines passing through a parts of the surface which are parallel

to the electric field (since $d\vec{A}$ is then perpendicular to the field), and a maximum when they are perpendicular to the surface.

Consider the case of a closed surface. That means that all the electric field lines originating on a charge (and that do not sink at an opposite charge) must eventually pass through the surface. The simplest example is to take a single charge, q , and consider a spherical surface centred on the charge and at a distance r . Since the electric field points directly outwards, it will always be perpendicular to the surface at every point, and so parallel to the surface normal, so we expect that $\vec{E} \cdot d\vec{A}$ will evaluate to $|\vec{E}||d\vec{A}|$. More explicitly, the surface element at a distance of r in spherical polar co-ordinates is:

$$d\vec{A} = r^2 \sin \theta d\theta d\phi \hat{r} \quad (3)$$

(Look back in the review of polar co-ordinates if this is unclear).

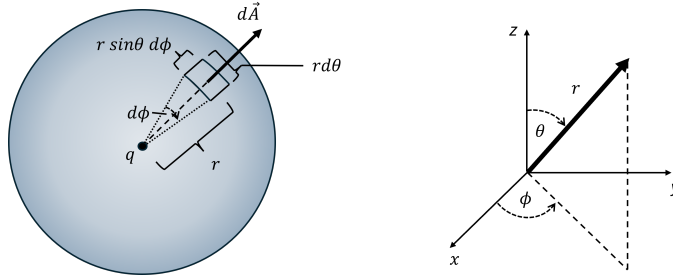


Figure 1: Calculation of flux through a spherical surface from an enclosed charge.

And so the flux through the spherical surface is:

$$\phi_E = \oint_S \vec{E} \cdot d\vec{A} = \frac{1}{4\pi\epsilon_0} q \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{1}{|\vec{r}|^2} r^2 \sin \theta d\theta d\phi \quad (4)$$

Notice we have used $\oint$ rather than $\int$ here, which is to remind us that this is a **closed surface**.

Since $|\vec{r}| = r$ this simplifies, and we can perform the integration:

$$\phi_E = \frac{q}{4\pi\epsilon_0} \int_0^{\pi} \int_0^{2\pi} \sin \theta d\theta d\phi \quad (5)$$

$$= \frac{q}{4\pi\epsilon_0} \int_0^{2\pi} (2\pi) \sin \theta d\theta \quad (6)$$

$$= \frac{q}{2\epsilon_0} \left(-\cos \theta \right) \Big|_0^{\pi} \quad (7)$$

$$\phi_E = \frac{q}{\epsilon_0} \quad (8)$$

Notice the lack of an r dependence. Regardless of how far out the spherical surface is from the charge, the same number of field lines pass through it; the flux is the same.

2 Gauss's Law

Now comes the important realisation. Above we showed that the flux through a spherical surface surrounding a charge q is q/ϵ_0 . But regardless of how we draw the surface the same number of field lines must eventually pass through the surface. So, we will have the same flux through any surface we can draw that encloses the charge. This is **Gauss's Law (in integral form)**:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \quad (9)$$

The surface is often called a **Gaussian Surface**. It does not have to be a real surface (i.e. there is nothing physically there), it is something we define.

We can write Gauss's Law in another way using the divergence theorem which relates the surface integral of a vector field to a volume integral of the divergence of the field,

$$\oint_S \vec{E} \cdot d\vec{A} = \int_V \nabla \cdot \vec{E} dV. \quad (10)$$

The total charge enclosed, q is the charged density ρ integrated over the enclosed volume (conceptually this is just 'adding up' all the charges inside the surface, so,

$$q = \int_V \rho dV \rightarrow \frac{q}{\epsilon_0} = \int_V \frac{\rho}{\epsilon_0} dV. \quad (11)$$

We therefore see that we have,

$$\int_V \frac{\rho}{\epsilon_0} dV = \int_V \nabla \cdot \vec{E} dV, \quad (12)$$

and hence:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}. \quad (13)$$

This is the **differential form of Gauss's Law**.

3 Interpreting Gauss's Law

Gauss's law in integral form has an interpretation that was obvious from the way we obtained it. It says that the flux through a closed surface (i.e. the field lines passing through a surface) depends only on the total charges enclosed by the surface, and not the arrangement of the charges or the shape of the surface.

In the differential form, we haven't mentioned a surface at all. This now telling us about divergence of a vector field at any point. The divergence tells us the difference between the field lines flowing into and out of that point. So anywhere where we are not sat on a charge, $\nabla \cdot \vec{E}$ is zero because field lines are not being created or destroyed, they are just passing through the point - what goes in also comes out. The divergence is not zero when we are on a charge, or in a region where the charge density ϕ is not zero, because there we have field line sources or sinks.

4 Using Gauss's Law

Gauss's Law is always true. However, there are only a few situations where it allows you to calculate anything useful analytically. Normally we will want to use it to calculate the electric field, $\vec{E}$. This is inside the integral on the left hand side, and we need to be able to take it outside the integral. This is only possible when we are able to define a surface such that the surface normal is always parallel to the direction of the electric field, and the magnitude of the electric field is the same everywhere on the surface. In that case, $\vec{E} \cdot d\vec{A} = |\vec{E}| \cdot |d\vec{A}|$ and E can be taken outside the integral.

As an example, let's consider a point charge, and try to compute the electric field at some distance, r . (Of course we know the answer, but let's see if we can get it from Gauss's Law).

We choose the Gaussian surface as a sphere centred on the charge. Since the electric field must be pointing directly outwards, by symmetry, it will then always be parallel to the surface normal, i.e. $\vec{E} \cdot d\vec{A} = E dA$, as shown in figure 2.

So we have,

$$\oint_S E dA = \frac{q}{\epsilon_0} \quad (14)$$

The element dA of the surface of a sphere in spherical polar co-ordinates is

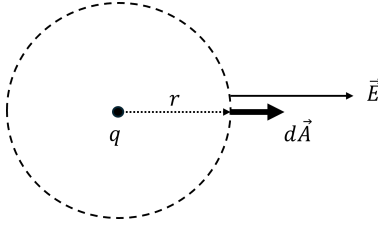


Figure 2: Gaussian surface for calculating field from point charge.

$dA = r^2 \sin \theta d\theta d\phi$. Integrate over the surface:

$$\oint_S E dA = E \int_0^{2\pi} \int_0^\pi r^2 \sin \theta d\theta d\phi \quad (15)$$

$$= E \int_0^\pi r^2 \sin \theta d\theta \phi \bigg|_{\phi=0}^{\phi=2\pi} \quad (16)$$

$$= E \int_0^\pi 2\pi r^2 \sin \theta d\theta \quad (17)$$

$$= -2\pi E r^2 \cos \theta \bigg|_{\theta=0}^{\theta=\pi} \quad (18)$$

$$= -2\pi E r^2 [-1 - 1] = 4\pi E r^2 \quad (19)$$

I have shown this for completeness, but you will see that this is simply E multiplied by the area of the surface, which will always be the case, so you can jump straight to this expression.

Then Gauss's Law says:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \quad (20)$$

$$4\pi E r^2 = \frac{q}{\epsilon_0} \quad (21)$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \quad (22)$$

This is the magnitude of the field, and we know that it must point in the $\hat{r}$ direction. So we have recovered the expression for the field from a point charge.

4.1 Field inside and outside a conducting sphere

If we have a solid, conducting sphere, then the electric field inside the sphere must be zero (otherwise the charges would move around until it is zero).

Gauss's law tells us that the field depends on the enclosed charge. We can draw a spherical surface inside the sphere (centered on the centre of the sphere) and the flux through the surface must be zero (since the field is zero everywhere in a conductor), i.e. the LHS of Gauss's Law is zero. This means that the RHS of Gauss's Law must also be zero, i.e. $q = 0$. So, there is no charge inside the sphere. If we put a charge on a conducting sphere, then it all must sit on the surface.

Now consider the field outside the sphere. Gauss's Law tells us that only the enclosed charge matters, not whether it is in the centre or the outside edge of the sphere. So the field is exactly the same as if the charge was all at the centre of the sphere.

Note this is **only works because the sphere is spherically symmetric**. If we had an irregular shape, then the field would depend on the distribution of charge. Why? Because now we would not have a simple expression on the LHS where the field is the same everywhere on the Gaussian surface, and we could not take E out of the integral. In this case, Gauss's Law would no longer be **useful** for us as a means of calculating E , although it would still be **true**.

4.2 Field inside and outside an insulating sphere

Now let's think about an insulated sphere, where the charge is not free to move around and so maintains a charge distribution. If the charge is uniformly distributed then we can use Gauss's Law quite easily, since the situation remains symmetrical, as shown in figure 3(a).

If the sphere has radius R and charge density ρ , then a Gaussian spherical surface of radius r has a surface area of $4\pi r^2$ and encloses a charge of $(4/3)\pi r^3 \rho$. Gauss's Law then gives:

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \quad (23)$$

$$(E)(4\pi r^2) = \frac{4\pi r^3 \rho}{3\epsilon_0} \quad (24)$$

$$E = \frac{r\rho}{3\epsilon_0} \quad (25)$$

and points radially outwards. The magnitude of the field is plotted in figure 3(b).

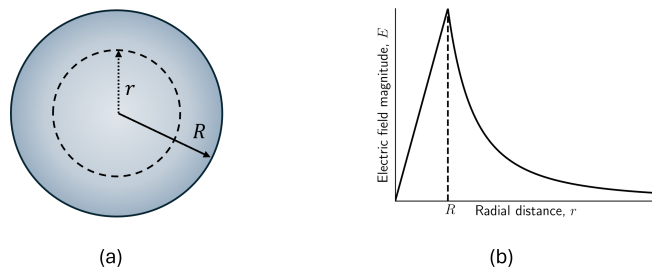


Figure 3: (a) Calculation of internal field inside sphere of charge; (b) plot of E-field as a function of distance from centre of sphere.

4.3 Field from a rod of charge

Another example is the field from a rod of charge. Recall we previously calculated this by directly integrating over the rod and solving a fairly tedious integral.

We choose a Gaussian surface which is the surface of a cylinder, of length l and radius r , with its axis parallel to the rod, as shown.

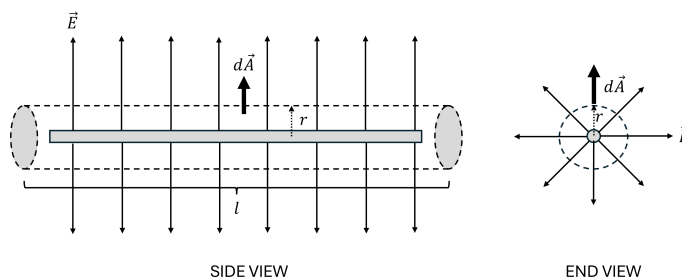


Figure 4: Calculation of field from rod of charge, showing cylindrical Gaussian surface.

The key thing to realise is that the electric field is perpendicular to the two ends of the cylinder (exactly for an infinitely long rod, and approximately true for a long rod, $l \gg r$.) So the contribution to $\vec{E} \cdot d\vec{A}$ from the end is zero and we only care about the curved side of the cylinder. Here the E-field is always perpendicular to the surface, so $\vec{E} \cdot d\vec{A} = E dA$.

The surface of the curved side of a cylinder is $2\pi r l$ where r is the radius and l is the length. The area of surface element dA is $dl r d\theta$, and the charge enclosed by the cylinder is λl . So we have

$$\begin{aligned}
\int E \cdot dA &= \frac{q}{\epsilon_0} \\
E \int_0^{2\pi} \int_0^l dl d\theta &= \frac{q}{\epsilon_0} \\
2\pi E r l &= \frac{\lambda l}{\epsilon_0} \\
E &= \frac{1}{2\pi\epsilon_0} \frac{\lambda}{r}
\end{aligned}$$

and we know this points radially, so

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{r} \hat{r},$$

which is exactly the expression we obtained previously, but with a much easier integral!

But that's not all. Gauss's Law is even more powerful. Notice that it says that the **flux through a closed surface only depends on the charge enclosed by the surface**. Charge outside the surface does not affect the total flux. (Spend some time thinking why this is the case - remember that it is a closed surface - what goes in must come out somewhere else!)

So let's suppose we have a rod with some radius R and uniform charge throughout (this must mean that it is an insulator), with volume charge density ρ . Suppose we would like to calculate the electric field inside the rod, i.e. at some radius $r < R$, as shown in figure 5(a). This is quite complicated to do with the integral method we used previously (although possible, you can give it a try).

But let's use Gauss's Law. The left hand side is the same as before, $2\pi E r l$.

The right hand side is different because the enclosed charge is now $\pi r^2 l \rho$.

Using Gauss's law we then find:

$$E = \frac{\rho r}{2\epsilon_0}$$

If we want this in terms of the total charge, q on a rod of length l , then we use that

$$\rho = \frac{q}{\pi R^2 l}$$

and substitute in:

$$E = \frac{1}{2\pi\epsilon_0} \frac{qr}{lR^2}$$

Notice that the field *increases* with r because we are then enclosing more and more of the total charge on the cylinder. Once we get outside the cylinder, $r > R$, then we revert back to the $1/r$ relationship. This is shown in figure 5(b) below.

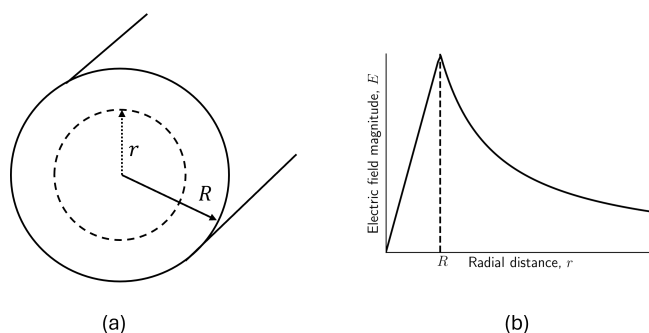


Figure 5: (a) Calculation of internal field inside rod of charge; (b) plot of E-field as a function of distance from centre of rod.

4.4 Field from a plane of charge

We also previously found that the magnitude of the field from a plane of charge is $E = \sigma/(2\epsilon_0)$ where σ is the surface charge density. Again we can do this using Gauss' law and again we choose a cylindrical surface, but now with the end of the cylinder parallel to the surface, as shown in figure 6.

We know the field lines are perpendicular to the surface, so there will be no field lines passing through the curved surface, only the top and bottom of the cylinder. The LHS of Gauss's Law is:

$$\oint \vec{E} \cdot d\vec{A} = (2)\pi r^2 \quad (26)$$

where the factor of 2 comes from the fact that we have both a top and bottom surface. I have jumped straight to the area of the top/bottom of the cylinder

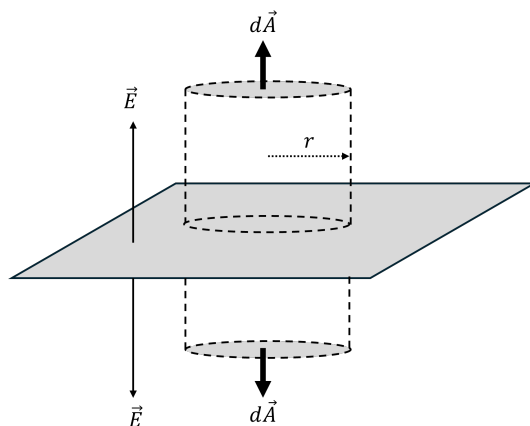


Figure 6: Gaussian surface for calculating field from plane of charge.

(πr^2) but you could explicitly do the integral:

$$\int_0^{2\pi} \int_0^r r \, dr \, d\theta = \pi r^2 \quad (27)$$

Now substituting into Gauss's Law, using that the charge on the surface covered by the cylinder is $\sigma\pi r^2$:

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \quad (28)$$

$$E(2\pi r^2) = \frac{\sigma\pi r^2}{\epsilon_0} \quad (29)$$

$$E = \frac{\sigma}{2\epsilon_0} \quad (30)$$

which immediately gives us the correct magnitude of the field, and we know it points perpendicular to the plane. This was a lot easier than the full integration.