Electromagnetism Part 6: Dipoles

Mike Hughes

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1 Introduction

Two opposite charges, in close proximity, considered together as a system, are called a **dipole**. Dipoles are important for the study of dielectrics in the next section, but are also a very useful practical tool for approximating more complex charge distributions.

2 Dipole moment

Consider two equal and opposite charges, $-q$ and $+q$, separate by a distance d, forming a dipole. We define the magnitude of the dipole moment as,

$$\rho = qd. \tag{1}$$

We can also write this as a vector, $\vec{\mu}$, which points from negative to positive charge.

Now we place the dipole in a uniform electric field. Two two charges are fixed relative to each other, but the dipole as a whole is free to rotate. The two poles of the dipole have opposite charges, and so experience a Coulomb force in opposite directions. So the dipole will experience a torque, and if free to rotate would align itself with the electric field.

The torque is,

Figure 1: Left: Electric dipole, Right: dipole in electric field, experiencing a torque that tends to align it with the field.

$$\vec{\tau} = \vec{\mu} \times \vec{E}, \tag{2}$$

or the magnitude of the torque,

$$\tau = \mu E \sin\theta, \tag{3}$$

where θ is the angle between the electric field and the dipole. If the dipole is aligned with the field then $\theta = 0$ and there is no torque, as we expect.

If the dipole is not aligned with the field then it has some potential energy, given by,

$$U = -\vec{\mu} \cdot \vec{E} = \mu E \cos\theta. \tag{4}$$

3 Electric field from a dipole

We know the electric field from a single charge drops off as $1/r^2$. A long way from a dipole it looks approximately like an electrically neutral point, and so we would expect the field to drop off faster.

We can show this in the perpendicular direction as follows:

If we have an electric dipole consisting of two point charges:

$$+q \text{ at } z = +a, \qquad -q \text{ at } z = -a$$

along the z-axis, then the dipole moment is,

$$\vec{\mu} = q(2a)\,\hat{z},$$

since $p = qd$ and $d = 2a$.

We will find the electric field at a point on the z-axis at position $z = r$, assuming,

$$r \gg a$$

For the positive charge, the distance to the field point is $(r - a)$, so,

$$E_+ = \frac{1}{4\pi\varepsilon_0} \frac{q}{(r-a)^2}$$

And for the negative charge the distance is $(r + a)$, giving

$$E_- = \frac{1}{4\pi\varepsilon_0} \frac{q}{(r+a)^2}$$

The fields are opposite, so the total field is,

$$E = \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{(r-a)^2} - \frac{q}{(r+a)^2} \right)$$

Using the binomial approximation for $r \gg a$,

$$\frac{1}{(r \pm a)^2} \approx \frac{1}{r^2} \left(1 \mp \frac{2a}{r} \right)$$

Substituting into the expression for E,

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \left[\left(1 + \frac{2a}{r} \right) - \left(1 - \frac{2a}{r} \right) \right]$$

Simplifying,

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \left(\frac{4a}{r} \right) = \frac{1}{4\pi\varepsilon_0} \frac{4qa}{r^3}$$

Since,

$$p = 2qa,$$

we obtain,

$$E = \frac{1}{4\pi\varepsilon_0} \frac{2p}{r^3}.$$

showing the $1/r^3$ dependence.

4 Potential from a Dipole

The potential from a dipole along the direction perpendicular to the axis is zero because at each point the potential from the two charges cancel. More generally, in any direction, it is,

$$V_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{\mu} \cdot \hat{r}}{|\vec{r}|^2}, \tag{5}$$

where unit vector $\hat{r}$ point from the dipole to the point where we calculate V. It is an exercise on the problem sheet to show this, make sure you can do it.

In the perpendicular direction, $\vec{\mu} \cdot \hat{r} = 0$, and so $V = 0$ as expected, whereas parallel to the dipole moment it is $|\vec{\mu}| = \rho$, in which case,

$$V = \frac{1}{4\pi\epsilon_0}\frac{\rho}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{qd}{r^2}.$$

Note that the potential drops off as $1/r^2$ in the perpendicular direction rather than $1/r$ for a point charge.

5 Multipole Expansion

In electrostatics, we can always work out the electric potential (in principle) at $\vec{r}$ produced by a charge distribution $\rho(\vec{r}')$ by integration, i.e. using,

$$V\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}\,d\tau'.$$

This expression is exact, but it is usually impossible to solve analytically unless the charge distribution has a high degree of symmetry. In many situations of interest, however, we are only concerned with the potential far away from the charges. In this case, the potential can be approximated using a **multipole expansion**, a series of terms which fall off with increasing power of $1/r$.

5.1 Monopole Term

The first term in the multipole expansion depends only on the total charge,

$$Q = \int \rho(\vec{r}')\,d\tau'.$$

At large distances, the potential reduces to that of a point charge,

$$V_{\text{mono}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}.$$

This is essentially saying that if we go far enough away, a charged object starts to look like a point charge. We have seen this before. However, what if there is some distribution of charge within an object, but the total charge is zero. Then, all this tells us is that if we go far enough way the potential tends to zero, but that's not very interesting!

5.2 Dipole Term

If the total charge is zero, the monopole term vanishes and the leading remaining contribution is the **dipole term**. The electric dipole moment of a charge distribution is defined as

$$\vec{\mu} = \int \vec{r}'\,\rho(\vec{r}')\,d\tau'.$$

i.e., we are taking a vector integral over infinitesimal dipoles. The dipole contribution to the potential at large distances is something we have already seen, i.e.,

$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{\mu}\cdot\hat{r}}{r^2},$$

where $\hat{r} = \vec{r}/r$ is the unit vector pointing from the charge distribution to the observation point.

As we saw, the dipole potential falls off as $1/r^2$, faster than the monopole term, and depends on the orientation of the dipole relative to the observation point.

5.3 Higher Multipoles

If both the total charge and the dipole moment vanish, the leading contribution comes from the **quadrupole term**, which falls off as $1/r^3$. Higher-order terms (octupole, hexadecapole, etc.) fall off even more rapidly with distance and encode increasingly fine details of the charge distribution.

In practice, the monopole and dipole terms are usually sufficient to describe the potential far from a charge distribution.

5.4 Example: Electric Dipole

Consider an electric dipole consisting of two point charges, $+q$ and $-q$, separated by a small distance d. The dipole moment is

$$\vec{p} = q\,\vec{d},$$

where $\vec{d}$ points from the negative to the positive charge.

The monopole term vanishes since the total charge is zero. The leading contribution to the potential is therefore the dipole term,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}.$$

If θ is the angle between $\vec{p}$ and $\vec{r}$, this may be written as

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}.$$

This expression shows explicitly that the dipole potential depends on both the distance from the dipole and its orientation. Along the dipole axis ($\theta = 0$), the potential is maximised, while in the plane perpendicular to the dipole ($\theta = \pi/2$), the potential vanishes.